1. Let $V$ and $W$ be Hilbert spaces, and let $\mathcal{B} : V \to W'$ be continuous and linear. Define the adjoint operator $\mathcal{B}' : W \to V'$ by $\mathcal{B}'w(v) = \mathcal{B}v(w)$, $\forall v \in V$, $w \in W$. Show that $\mathcal{B}'$ is continuous and that its adjoint is given by $\mathcal{B}'' = \mathcal{B}$.

2. Let $U$ be a subset of $W$. The annihilator of $U$ is the set of functionals given by

$$U^\circ = \{ f \in W' : f(w) = 0 \ \forall w \in U \}.$$ 

Show that $U^\circ$ is a closed subspace of $W'$.

3. Let the continuous and linear operator $\mathcal{B} : V \to W'$ be given as above.

(a) Show that the closure of the range of $\mathcal{B}$ is the annihilator of the kernel of $\mathcal{B}'$, that is,

$$\overline{\mathcal{B}(V)} = (\text{Ker}\mathcal{B}')^\circ.$$ 

(b) If there exists an $\alpha > 0$ such that

$$\sup_{w \in W} \frac{\mathcal{B}v(w)}{\|w\|_W} \geq \alpha \|v\|_V \ \forall v \in V;$$

show that $\mathcal{B}$ is an isomorphism of $V$ onto $(\text{Ker}\mathcal{B}')^\circ$.

4. Let $V$ be a Hilbert space, and let $\mathcal{A} : V \to V'$ be symmetric and continuous. Let $W$ be another Hilbert space and $\mathcal{B} : V \to W'$ be continuous and linear. Assume that $\mathcal{A}$ is Ker$\mathcal{B}$-coercive.

(a) Let $f \in V'$ and $g \in \mathcal{B}(V) \subset W'$ be given. Show that there exists a unique

$$u \in V : \mathcal{B}(u) = g \quad \text{and} \quad \mathcal{A}(u) - f \in (\text{Ker}\mathcal{B})^\circ.$$ 

(b) If additionally the range of $\mathcal{B}'$ is closed, then there exists a unique solution of the mixed formulation, a pair

$$(u, \lambda) \in V \times W : \mathcal{B}(u) = g \quad \text{and} \quad \mathcal{A}(u) + \mathcal{B}'(\lambda) = f.$$ \hspace{1cm} (1)$$

5. Choose the spaces and operators as follows:

$$V = H^1(a, b), \quad \mathcal{A}u(v) = \int_a^b uu'dx,$$

$$W = \mathbb{R}, \quad \mathcal{B}(u) = u(c), \quad \text{where} \ a < c < b,$$

$$f = f_a + f_b + F, \quad \text{where} \ f_a, f_b \in \mathbb{R}, \ F \in L^2(a, b), g \in \mathbb{R}.$$ 

(a) Show that $\mathcal{B}'(1) = \delta_c$.

(b) Show that (1) has exactly one solution.

(c) Show that the solution of (1) satisfies the system

$$-u'' + \lambda \delta_c = F \text{ in } C_0^\infty(a, b)^*;$$

$$-u'(a) = f_a, \quad u'(b) = f_b, \quad u(c) = g.$$