Algebra - hw # 1

fall 2008

Some general comments - A few things to keep in mind

• Some of you seem to have good ideas but some trouble expressing them. Welcome to graduate school: it's time to improve your style. Keep in mind that the grader is only a human being just like yourself, therefore not a mind reader. The grader can only grade what you write on the paper. Even more, the grader can only grade that part that is readable of what you write on the paper. And I don't think I would be helping you if I don't let you know when I find what you write unreadable, either for reasoning or for handwriting issues. Remember that when you submit a paper for publishing, the referees wont' necessarily be kind or merciful. So here's your training. You are welcome.

• Examples are good to see patterns and understand how things work, but they do NOT prove things. (They might however disprove things, in the right context).

• No work = no credit. Regardless of wether you got the answer right or wrong. If you don't show how you got your answer, the grader will systematically assume that you copied it from some suspicious source, other than your own work.

1. The Petersen graph.

Some of you showed that the automorphism group G has 120 elements (which is correct), and argued that it was then enough to prove that S_5 acts on G. This seems to assume the fact that there is a unique group of order 120 acting on G. Next time you might want to say a few words about this.

Some of you constructed a map between $Aut(\Gamma)$ and S_5 and proved it an isomorphism. You might want to check if perhaps the same ideas you used to proof surjectivity wouldn't have been enough to solve the whole problem at once.

2. A semidirect product.

If you say that there are "exactly four" subgroups of $(\mathbb{Z}/2)^n$ that are normal in G then you must specify for what n this is true. The assertion is false for some values of n.

If you find four subgroups that are normal that's very nice, but you still need to explain why you don't have anything else.

4. A non-obvious group action

This exercise seems to have been the one where most of you had more problems with. Here are some **rough ideas** (this is not a solution).

(a) This problem "smells" a lot like problem number 1: we had S_5 acting not on 5 elements but on something different. Here we are asked to do something kind of like that with S_6. So it might be worthy to see if using some of those ideas could be helpful.

Let $\Gamma = (V, E)$ be the graph whose vertices are the *triplets* (=subsets of size 3) of $\{1, 2, 3, 4, 5, 6\}$, with two vertices joined if the corresponding triplets are disjoint. (Sounds familiar?) A big difference though is that here for each triplet $t \in V$ there is a unique choice of t^c, therefore each vertex is adjacent to exactly one edge, giving us a disconnected graph on 10 edges:

$$
\begin{array}{lcr}
\{1,2,3\} \bullet & \text{———} & \bullet \{4,5,6\} \\
\{1,2,4\} \bullet & \text{———} & \bullet \{3,5,6\} \\
\{1,2,5\} \bullet & \text{———} & \bullet \{3,4,6\} \\
\{1,2,6\} \bullet & \text{———} & \bullet \{3,4,5\} \\
\{1,3,4\} \bullet & \text{———} & \bullet \{2,5,6\} \\
\{1,3,5\} \bullet & \text{———} & \bullet \{2,4,6\} \\
\{1,3,6\} \bullet & \text{———} & \bullet \{2,4,5\} \\
\{1,4,5\} \bullet & \text{———} & \bullet \{2,3,6\} \\
\{1,4,6\} \bullet & \text{———} & \bullet \{2,3,5\} \\
\{1,5,6\} \bullet & \text{———} & \bullet \{2,3,4\}
\end{array}
$$

From now on you can follow the same line of reasoning you used for the Peterson graph and finish part (a).

(b) Point stabilizer

Let B be a pair of disjoints triplets t and t^c (we call such an B a *bisection*). The necessary and sufficient condition for any $\sigma \in S_6$ to belong in the point stabilizer G_B is

$$\sigma t = t \qquad \text{or} \qquad \sigma t = t^c$$

where σt means σ applied to the *set* t. Hope this helps.

(c) S_6 acts 2-transitively on the set of bisections

It obviously acts transitively, so it remains to show the stabilizer of any given bisection acts transitively on the others. For example, let

$$B = \{123, 456\}$$

The stabilizer of B acts transitively on the size 3 sets other than $\{123\}$ and $\{456\}$. To see this, we will show that any one, say $\{rst\}$, can be sent to $\{124\}$. I'll drop the brackets now.

Now, rst intersects one of 123 and 456 in two points. By swapping 123 with 456 if necessary, which preserves B (!), we may suppose wlog that rst meets 123 in two points and 456 in one. Now apply an element of $S_3 \times S_3$ sending the two points in 123 to 1 and 2 and the one point in 456 to 4.