Optimal investment with high-watermark performance fee

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based on joint work with

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Objective

- build and analyze a model of optimal investment and consumption where the investment opportunity is represented by a hedge-fund using the "two-and-twenty rule"
- analyze the impact of the high-watermark fee on the investor
Previous work on hedge-funds and high-watermarks

All existing work analyzes the impact/incentive of the high-watermark fees on **fund managers**

- extensive finance literature
  - Goetzmann, Ingersoll and Ross, Journal of Finance 2003
  - Panagea and Westerfield, Journal of Finance 2009
  - Agarwal, Daniel and Naik Journal of Finance, forthcoming
  - Aragon and Qian, preprint 2007

- recently studied in mathematical finance
  - Guasoni and Obloj, preprint 2009
A model of profits from dynamically investing in a hedge-fund

- the investor chooses to hold $\theta_t$ in the fund at time $t$
- the value of the fund $F_t$ is given \textit{exogenously}
- denote by $P_t$ the accumulated profit/losses up to time $t$

Evolution of the profit

- without high-watermark fee
  \[ dP_t = \theta_t \frac{dF_t}{F_t}, \quad P_0 = 0 \]

- with high-watermark proportional fee $\lambda > 0$
  \[
  \begin{cases}
  dP_t = \theta_t \frac{dF_t}{F_t} - \lambda dP^*_t, & P_0 = 0 \\
  P^*_t = \max_{0 \leq s \leq t} P_s
  \end{cases}
  \]

High-watermark of the investor

\[ P^*_t = \max_{0 \leq s \leq t} P_s. \]

Observation: can be also interpreted as taxes on gains, paid right when gains are realized (pointed out by Paolo Guasoni)
Path-wise solutions

(same as Guasoni and Obloj)
Denote by $I_t$ the paper profits from investing in the fund

$$I_t = \int_0^t \theta_u \frac{dF_u}{F_u}$$

Then

$$P_t = I_t - \frac{\lambda}{\lambda + 1} \max_{0 \leq s \leq t} I_s$$

The high-watermark of the investor is

$$P^*_t = \frac{1}{\lambda + 1} \max_{0 \leq s \leq t} I_s$$

Observations:

- the fee $\lambda$ can exceed 100% and the investor can still make a profit
- the high-watermark is measured before the fee is paid
Connection to the Skorohod map (Part of work in progress with Gerard Brunick)

Denote by $Y = P^* - P$ the distance from paying fees. Then $Y$ satisfies the equation:

$$\begin{cases}
    dY_t = -\theta_t \frac{dF_t}{F_t} + (1 + \lambda) dP_t^* \\
    Y_0 = 0,
\end{cases}$$

where $Y \geq 0$ and

$$\int_0^t \mathbb{I}\{Y_s \neq 0\} dP^*_s = 0, \quad (\forall) \ t \geq 0.$$

Skorohod map

$$I. = \int_0^\cdot \theta_u \frac{dF_u}{F_u} \rightarrow (Y, P^*) \approx (P, P^*).$$

Remark: $Y$ will be chosen as state in more general models.
The model of investment and consumption

An investor with initial capital $x > 0$ chooses to

- have $\theta_t$ in the fund at time $t$
- consume at a rate $\gamma_t$
- finance from borrowing/investing in the money market at zero rate

Denote by $C_t = \int_0^t \gamma_s ds$ the accumulated consumption. Since the money market pays zero interest, then

$$X_t = x + P_t - C_t \iff P_t = (X_t + C_t) - x$$

Therefore, the fees (high-watermark) is computed tracking the wealth and accumulated consumption

$$P_t^* = \max_{0 \leq s \leq t} \left\{ X_s + \int_0^s \gamma_u du \right\} - x$$

Can think that the investor leaves all her wealth (including the money market) with the investor manager.
Evolution equation for the wealth

The evolution of the wealth is

\[
\begin{align*}
\left\{
\begin{array}{l}
dX_t &= \theta_t \frac{dF_t}{F_t} - \gamma_t dt - \lambda dP_t^*, \\
P_t^* &= \max_{0 \leq s \leq t} \{ X_s + \int_0^s \gamma_u du \} - x
\end{array}
\right.
\end{align*}
\]

- Consumption is a part of the running-max, as opposed to the literature on draw-dawn constraints
  - Grossman and Zhou
  - Cvitanic and Karatzas
  - Elie and Touzi
  - Roche

- We still have a similar path-wise representation for the wealth in terms of the "paper profit" $l_t$ and the accumulated consumption
Admissible strategies

\[ \mathcal{A}(x) = \{(\theta, \gamma) : X > 0 \}. \]

Can represent investment and consumption strategies in terms of proportions

\[ c = \gamma / X, \quad \pi = \theta. \]

Observation:

- no closed form path-wise solutions for \( X \) in terms of \((\pi, c)\) (unless \( c = 0 \))
Maximize discounted utility from consumption on infinite horizon

\[ \mathcal{A}(x) \ni (\theta, \gamma) \rightarrow \arg\max \mathbb{E} \left[ \int_0^\infty e^{-\beta t} U(\gamma_t) dt \right]. \]

Where \( U : (0, \infty) \rightarrow \mathbb{R} \) is the CRRA utility

\[ U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \quad p > 0. \]

Finally, choose a geometric Brownian-Motion model for the fund share price

\[ \frac{dF_t}{F_t} = \alpha dt + \sigma dW_t. \]
Dynamic programming: state processes

Fees are paid when \( P = P^* \). This can be translated as 
\( X + C = (X + C)^* \) or as
\[
X = (X + C)^* - C.
\]

Denote by
\[
N \triangleq (X + C)^* - C.
\]

The (state) process \((X, N)\) is a two-dimensional controlled diffusion \( 0 < X \leq N \) with reflection on \( \{X = N\} \).

The evolution of the state \((X, N)\) is given by
\[
\begin{aligned}
dX_t &= (\theta_t \alpha - \gamma_t) \, dt + \theta_t \sigma \, dW_t - \lambda dP_t^*, \quad X_0 = x \\
dN_t &= -\gamma_t \, dt + dP_t^*, \quad N_0 = x.
\end{aligned}
\]

Recall we have path-wise solutions in terms of \((\theta, \gamma)\).
we are interested to solve the problem using dynamic programing. We are only interested in the initial condition \((x, n)\) for \(x = n\) but we actually solve the problem for all \(0 < x \leq n\). This amounts to setting an initial high-watemark of the investor which is larger than the initial wealth.

expect to find the two-dimensional value function \(v(x, n)\) as a solution of the HJB, and find the (feed-back) optimal controls.
Dynamic programming equation

Use Itô and write formally the HJB

\[
\sup_{\gamma \geq 0, \theta}\left\{ -\beta v + U(\gamma) + (\alpha \theta - \gamma)v_x + \frac{1}{2}\sigma^2 \theta^2 v_{xx} - \gamma v_n \right\} = 0
\]

for \(0 < x < n\) and the boundary condition

\[-\lambda v_x(x, x) + v_n(x, x) = 0.\]

(Formal) optimal controls

\[
\hat{\theta}(x, n) = -\frac{\alpha}{\sigma^2} \frac{v_x(x, n)}{v_{xx}(x, n)}
\]

\[
\hat{\gamma}(x, n) = I(v_x(x, n) + v_n(x, n))
\]
Denote by \( \tilde{U}(y) = \frac{p}{1-p} y^{\frac{p-1}{p}}, \quad y > 0 \) the dual function of the utility. The HJB becomes

\[
- \beta v + \tilde{U}(v_x + v_n) - \frac{1}{2} \frac{\alpha^2}{\sigma^2} \frac{v_x^2}{v_{xx}} = 0, \quad 0 < x < n
\]

plus the boundary condition

\[
- \lambda v_x(x, x) + v_n(x, x) = 0.
\]

Observation:

- if there were no \( v_n \) term in the HJB, we could solve it closed-form as in Roche or Elie-Touzi using the (dual) change of variable \( y = v_x(x, n) \)
- no closed-form solutions in our case (even for power utility)
Reduction to one-dimension

Since we are using power utility

\[ U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0 \]

we can reduce to one-dimension

\[ v(x, n) = x^{1-p} v\left(1, \frac{n}{x}\right) \]

and

\[ v(x, n) = n^{1-p} v\left(\frac{x}{n}, 1\right) \]

- first is nicer economically (since for \( \lambda = 0 \) we get a constant function \( v(1, \frac{n}{x}) \))
- the second gives a nicer ODE (works very well if there is a closed form solution, see Roche)

There is no closed form solution, so we can choose either one-dimensional reduction.
Reduction to one-dimension cont’d

We decide to denote \( z = \frac{n}{x} \geq 1 \) and

\[
v(x, n) = x^{1-p} u(z).
\]

Use

\[
v_n(x, n) = u'(z) \cdot x^{-p},
\]

\[
v_x(x, n) = \left( (1 - p)u(z) - zu'(z) \right) \cdot x^{-p},
\]

\[
v_{xx}(x, n) = \left( -p(1 - p)u(z) + 2pzu'(z) + z^2u''(z) \right) \cdot x^{-1-p},
\]

to get the reduced HJB

\[
-\beta u + \tilde{U}((1-p)u - (z-1)u') \frac{1}{\sigma^2} \frac{(1 - p)u - zu'}{2} \left( (1 - p)u + 2pzu' + z^2u'' \right) = 0
\]

for \( z > 1 \) with boundary condition

\[
-\lambda(1 - p)u(1) + (1 + \lambda)u'(1) = 0
\]
(Formal) optimal proportions

\[ \hat{\pi}(z) = \frac{\alpha}{p\sigma^2} \cdot \frac{(1 - p)u - zu'}{(1 - p)u - 2zu' - \frac{1}{p}z^2u''}, \]

\[ \hat{c}(z) = \left( \frac{v_x + v_n}{\chi} \right)^{-\frac{1}{p}} = \left( (1 - p)u - (z - 1)u' \right)^{-\frac{1}{p}} \]

Optimal amounts (controls)

\[ \hat{\theta}(x, n) = x\hat{\pi}(z), \quad \hat{\gamma}(x, n) = x\hat{c}(z) \]

Objective: solve the HJB analytically and then do verification
Solution of the HJB for $\lambda = 0$

This is the classical Merton problem. The optimal investment proportion is given by

$$\pi_0 \triangleq \frac{\alpha}{p \sigma^2},$$

while the value function equals

$$v_0(x, n) = \frac{1}{1 - p} c_0^{-p} x^{1-p}, \quad 0 < x \leq n,$$

where

$$c_0 \triangleq \frac{\beta}{p} - \frac{1}{2} \frac{1 - p}{p^2} \cdot \frac{\alpha^2}{\sigma^2}$$

is the optimal consumption proportion. It follows that the one-dimensional value function is constant

$$u_0(z) = \frac{1}{1 - p} c_0^{-p}, \quad z \geq 1.$$
Solution of the HJB for $\lambda > 0$

If $\lambda > 0$ we expect that (additional boundary condition)

$$\lim_{z \to \infty} u(z) = u_0.$$  

(For very large high-watermark, the investor gets almost the Merton expected utility)
Existence of a smooth solution

**Theorem 1** The HJB has a smooth solution.

Idea of solving the HJB:

- find a viscosity solution using an adaptation of Perron’s method. Consider infimum of concave supersolutions that satisfy the boundary condition. Obtain as a result a concave viscosity solution. The subsolution part is more delicate. Have to treat carefully the boundary condition.
Proof of existence: cont’d

- show that the viscosity solution is $C^2$ (actually more). Concavity, together with the subsolution property implies $C^1$ (no kinks). Go back into the ODE and formally rewrite it as

$$u'' = f(z, u(z), u'(z)) \triangleq g(z).$$

Compare locally the viscosity solution $u$ with the classical solution of a similar equation

$$w'' = g(z)$$

with the same boundary conditions, whenever $u, u'$ are such that $g$ is continuous. The difficulty is to show that $u, u'$ always satisfy this requirement.

Avoid defining the value function and proving the Dynamic Programming Principle.
Verification, Part I

**Theorem 2** The closed loop equation

\[
\begin{aligned}
    dX_t &= \hat{\theta}(X_t, N_t) \frac{dF_t}{F_t} - \hat{\gamma}(X_t, N_t) dt - \lambda(dN_t + \gamma_t dt), \quad X_0 = x \\
    N_t &= \max_{0 \leq s \leq t} \left\{ X_s + \int_0^s \hat{\gamma}(X_u, N_u) du \right\} - \int_0^t \hat{\gamma}(X_u, N_u) du
\end{aligned}
\]

has a unique strong solution $0 < \hat{X} \leq \hat{N}$.

Idea of proof:

- use the path-wise representation

\[
(Y, L) \rightarrow (\hat{\theta}(Y, L), \hat{\gamma}(Y, L)) \rightarrow (X, N)
\]

together with the Itô-Picard theory to obtain a unique global solution $X \leq N$.

- use the fact that the optimal proportion $\hat{\pi}$ and $\hat{c}$ are bounded to compare $\hat{X}$ to an exponential martingale and conclude

\[
\hat{X} > 0
\]
Theorem 3  The controls $\hat{\theta}(\hat{X}_t, \hat{N}_t)$ and $\hat{\gamma}(\hat{X}_t, \hat{N}_t)$ are optimal.

Idea of proof:

$\blacktriangleright$ use Itô together with the HJB to conclude that

$$e^{-\beta t} V(X_t, N_t) + \int_0^t e^{-\beta s} U(\gamma_s) ds, \quad 0 \leq t < \infty,$$

is a local supermartingale in general and a local martingale for the candidate optimal controls (the obvious part)

$\blacktriangleright$ uniform integrability. Has to be done separately for $p < 1$ and $p > 1$ (the harder part, requires again the use of $\hat{\pi}$ and $\hat{c}$ bounded, and comparison to an exponential martingale).
The impact of fees

Everything else being equal, the fees have the effect of

- reducing rate of return
- reducing initial wealth
Certainty equivalent return

We consider two investors having the same initial wealth, risk-aversion, who invest in two funds with the same volatility.

- one invests in a fund with return $\alpha$, and pays fees $\lambda > 0$. The initial high-watermark is $n = xz \geq x$.
- the other invests in a fund with return $\tilde{\alpha}$ but pays no fees.

Equate the expected utilities:

$$u_0(\tilde{\alpha}(z), \cdot) = u_\lambda(\alpha, z).$$

Can be solved as

$$\tilde{\alpha}^2(z) = 2\sigma^2 \frac{p^2}{1 - p} \left( \frac{\beta}{p} - ((1 - p) u_\lambda(z))^{-\frac{1}{p}} \right), \quad z \geq 1.$$

The relative size of the certainty equivalent excess return is therefore

$$\frac{\tilde{\alpha}(z)}{\alpha} = \sqrt{2\sigma p} \left( \frac{\beta}{p} - ((1 - p) u_\lambda(z))^{-\frac{1}{p}} \right)^{\frac{1}{2}}, \quad z \geq 1.$$
Certainty equivalent initial wealth

We consider two investors having the same risk-aversion, who invest in the same fund

- one has initial wealth $x$, initial high-watermark $n = xz \geq x$ and pays fees $\lambda > 0$
- the other has initial wealth $\tilde{x}$ but pays no fees

Equate the expected utilities:

$$\tilde{x}(z)^{1-p} u_0(\cdot) = v_0(\tilde{x}(z), \cdot) = v_\lambda(x, n) = x^{1-p} u_\lambda(z)$$

all other parameters being equal. Can be solved as

$$\tilde{x}(z) = x \cdot \left( \frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = x \cdot ((1 - p)c_0^p u_\lambda(z))^{\frac{1}{1-p}}, \quad z \geq 1.$$ 

The quantity

$$\frac{\tilde{x}(z)}{x} = \left( \frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = ((1 - p)c_0^p u_\lambda(z))^{\frac{1}{1-p}}, \quad z \geq 1,$$

is the relative certainty equivalent wealth.
Investment proportion relative to Merton proportion

X to N ratio

- $p = 3, \beta = 5\%, \alpha = 10\%, \lambda = 20\%$
- $p = 3, \beta = 5\%, \alpha = 20\%, \lambda = 20\%$
- $p = 3, \beta = 5\%, \alpha = 30\%, \lambda = 20\%$
- $p = 3, \beta = 5\%, \alpha = 10\%, \lambda = 40\%$
- $p = 3, \beta = 5\%, \alpha = 10\%, \lambda = 60\%$
- $p = 10, \beta = 5\%, \alpha = 10\%, \lambda = 20\%$
- $p = 3, \beta = 0\%, \alpha = 10\%, \lambda = 20\%$
Consumption proportion relative to Merton consumption

\[ p = 3, \beta = 5\%, \alpha = 10\%, \lambda = 20\% \]

\[ p = 3, \beta = 5\%, \alpha = 20\%, \lambda = 20\% \]

\[ p = 3, \beta = 5\%, \alpha = 30\%, \lambda = 20\% \]

\[ p = 3, \beta = 5\%, \alpha = 10\%, \lambda = 40\% \]

\[ p = 10, \beta = 5\%, \alpha = 10\%, \lambda = 20\% \]

\[ p = 3, \beta = 0\%, \alpha = 10\%, \lambda = 20\% \]
Relative certainty equivalence zero fee return

\begin{align*}
p = 3, & \quad \beta = 5\%, \quad \alpha = 10\%, \quad \lambda = 20\% \\
p = 3, & \quad \beta = 5\%, \quad \alpha = 20\%, \quad \lambda = 20\% \\
p = 3, & \quad \beta = 5\%, \quad \alpha = 30\%, \quad \lambda = 20\% \\
p = 3, & \quad \beta = 5\%, \quad \alpha = 10\%, \quad \lambda = 40\% \\
p = 3, & \quad \beta = 5\%, \quad \alpha = 10\%, \quad \lambda = 60\% \\
p = 10, & \quad \beta = 5\%, \quad \alpha = 10\%, \quad \lambda = 20\% \\
p = 3, & \quad \beta = 0\%, \quad \alpha = 10\%, \quad \lambda = 20\% 
\end{align*}
Conclusions

Point of view of Finance:

▶ model optimal investment with high-watermark fees from the point of view of the investor
▶ analyze the impact of the fees

Point of Mathematics:

▶ an example of controlling a two-dimensional reflected diffusion
▶ solve the problem using direct dynamic programming: first find a smooth solution of the HJB and then do verification

”Meta Conclusion”:

▶ whenever one can prove enough regularity for the viscosity solution to do verification, the viscosity solution can/should be constructed analytically, using Perron’s method, and avoiding DPP altogether
Work in progress and future work

with Gerard Brunick and Karel Janeček

- presence of (multiple and correlated) traded stocks, interest rates and hurdles: can still be modeled as a two-dimensional diffusion problem using $X$ and $Y = P - P^*$ as state processes (reduced to one-dimension by scaling)
- analytic approximations when $\lambda$ is small
- more than one fund: genuinely multi-dimensional problem with reflection
- stochastic volatility, jumps, etc
Where does it all go?

Investor

- can either invest in a number of assets \((S_1, \ldots, S_n)\) with transaction costs
- invest in the hedge-fund \(F\) paying profit fees.

The hedge-fund

- can invest in the assets with lower (even zero for mathematical reasons) transaction costs, and produce the fund process \(F\).

For certain choices of \(F\) (time-dependent combinations of the stocks and money market), one can compare the utility of the investor in the two situations: this should the existence of hedge-funds (from the point of view of the investor).

Actually, the whole situation should be modeled as a game between the investor and the hedge fund.