SMMG October 14th, 2006
featuring Dr. Michael Starbird
"Circles, Rings, and Tractors:
Clever Cleaving for Finding Formulas"

1. **The Polygon Sweep**

a) On one of the colored pieces of construction paper you find the image of a quadrilateral (four-sided polygon). Use the following guideline to do your own sweep-out:
   - extend all the sides of the polygon with a ruler (clockwise)
   - take a compass and open it to a length of 2 inches
   - from each vertex of the polygon, use the compass to draw the segment of a circle with radius 2 inches until you hit the next extended side of the polygon
   - Look at the pretty picture for a moment! What do you think the area of all those wedges you just drew is going to add up to?

   Now cut out those wedges and put them together - what do you get?

b) Next take the sheet with the two circles on it.

   - First choose six points on the upper circle and draw the inscribed hexagon that has those points as its vertices. Then go through the same procedure as in part b).

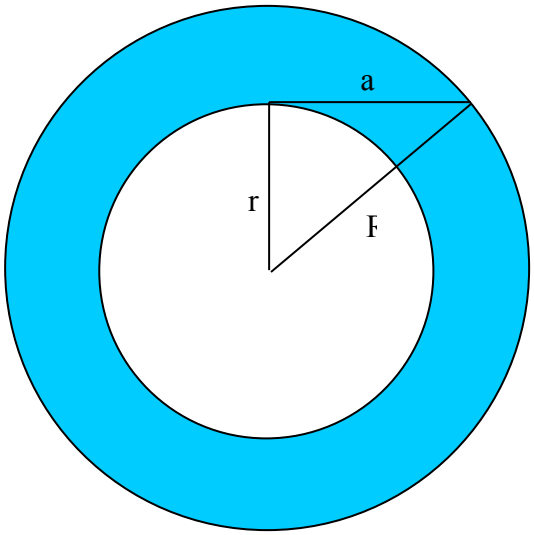
   - Choose twelve points on the other circle, draw the corresponding 12-sided polygon and then repeat the same steps again.

c) Imagine you were to choose more and more points on the circle so that the many sides approximate the circle as close as you want. Then what does the sweep-out picture start to look like?

   Make a conjecture:

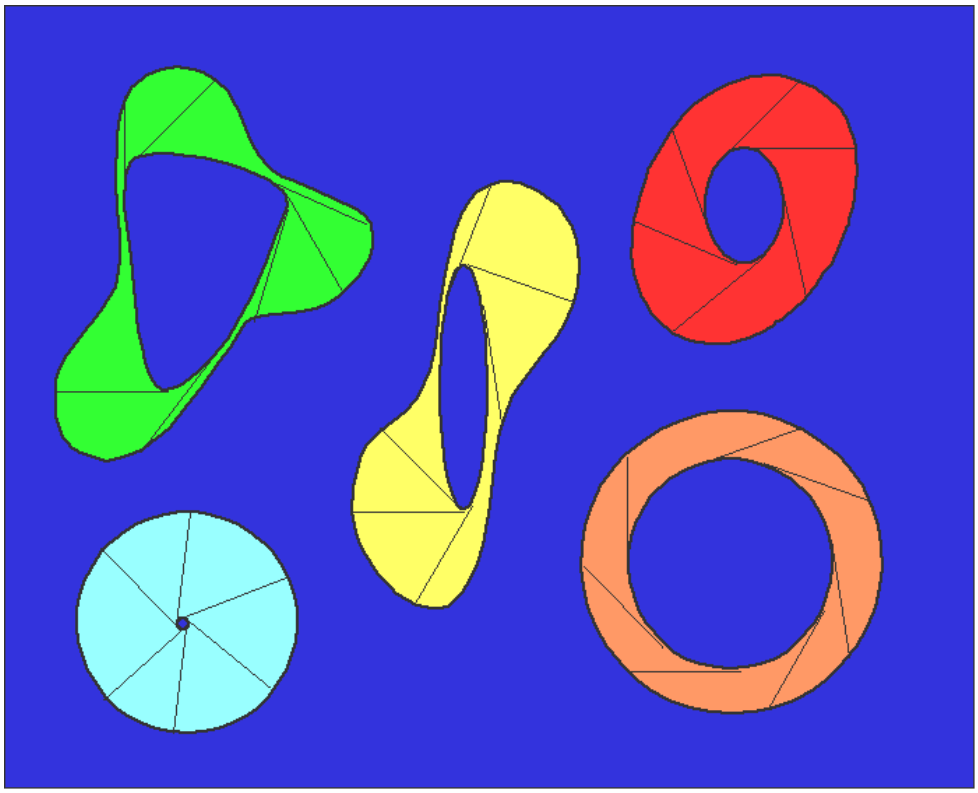
   The area of _____

   equals the area of _____ .



2. From Corners to Smoothies

a) Which cake is more? That is, which of the tangent sweep-outs has the biggest area?

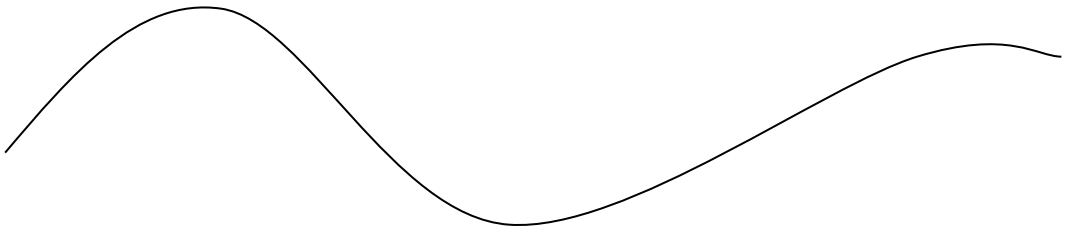


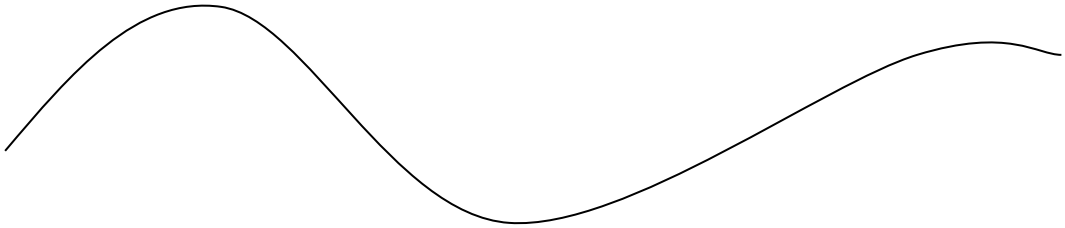
b) We'll be handing you some tools to make your own tangent sweep picture.

- Draw a nice smooth curve on the piece of cardstock (or copy the curve we suggest at the end of the handout).
- Pick at least 12 points on the curve.
- At each of those points, find the tangent to the curve and glue a toothpick along that tangent, with its end at the point you chose.
- Next, combine all the tips of the toothpicks to form another curve and voilà, you have created your own individual SMMG piece of art!

c) Below you see two identical curves. Imagine creating tangent sweeps by drawing tangential arrows of length 1 inch in such a way that once you have chosen a direction for the first curve, choose the tangential vector in the opposite direction for the second. Do you think that the two sweep-outs are going to be the same?

Now do the actual sweep-out and check your answer.

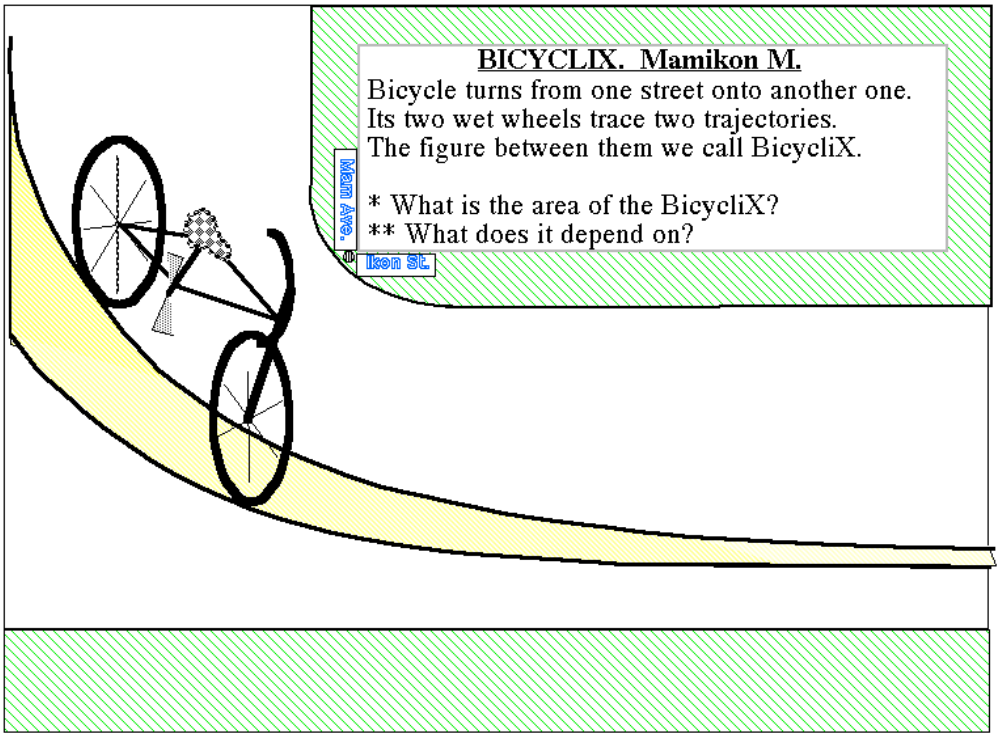




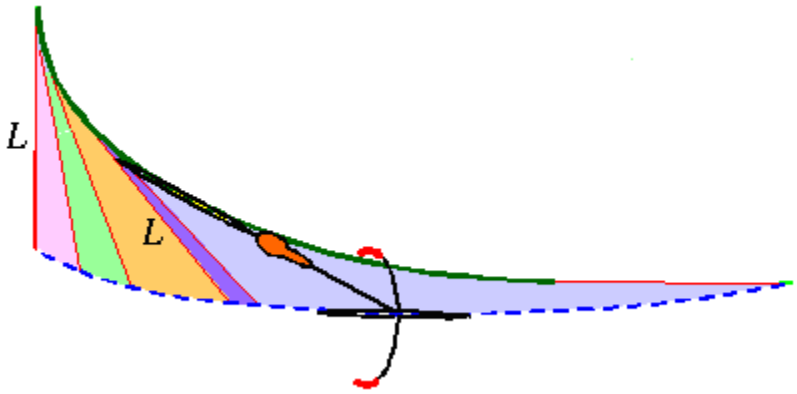
3. Tractors and Bicycles

a) The Bicyclix

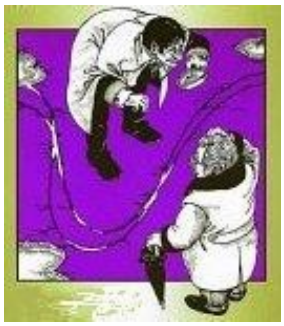
We have seen the tractrix, the trajectory of a toy on a taut string being pulled by a child walking in a straight line. A similar real-world situation occurs when a bicycle's front wheel traces out one curve while the rear wheel traces out another curve. The figure between them we call Bicyclix. Again, you are able to find the area between those two curves without using calculus!



Maybe the following picture will help you find the answer. Assume that the bike makes an exact 90 degree turn. Rearrange the wedges – what do you obtain? Why do all the wedges have the same length?



b) Mystery Story: Which way did the bicycle go?



Sherlock Holmes and Dr. Watson arrived sharp, 5:00 PM, at the scene of the crime. It was an old bank building, and there stood Old Man Johnson, the 60-some year old Banker.
[in the background] "He's done it to me again, I tell ya, that Billy The Kid's been out robbing me..."

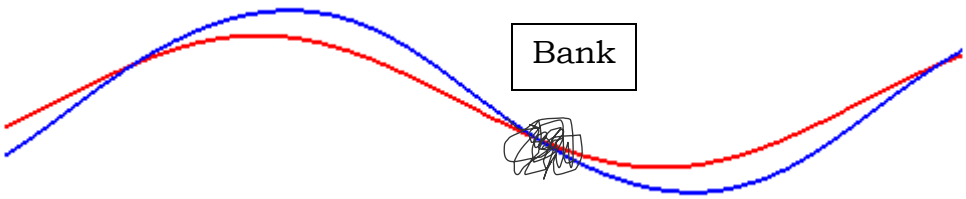
"Watson, look down in the sand, tell me what you see."
Watson took out his magnifying glass and replied, "I see millions and millions of of sand grains."
Holmes was silent for a minute, then spoke again. "Elementary, my dear Watson! Besides sand grains, here are Billy The Kid's bicycle tracks. We have our robber, if only we can figure out which way he went!"

Sherlock Holmes and Dr. Watson call for your help! With your knowledge of geometry that you learned today, you are able to find a nice geometrical solution to this problem. It is based on the following observations regarding a normal bicycle:

- From riding bikes, one may deduce that the front tire should oscillate more than the back tire – but this distinction is hard to make for the given tracks. In fact, it is not even possible to figure out which track crossed over on top of the other.
- The distance between the points of contact of the two tires and the ground is constant.
- The rear wheel always moves towards the front wheel as it is being pulled by the frame.

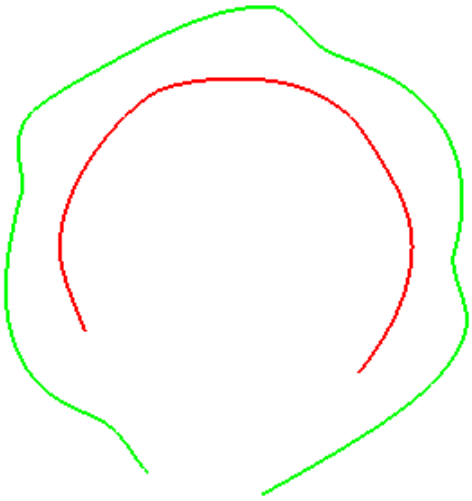
This implies that the tangent of the rear wheel track should intersect the front wheel track at a given distance that is unknown but constant for any given bicycle.

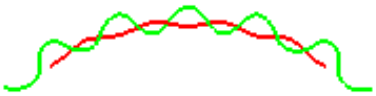
Go ahead and color the front wheel track in blue, the back wheel track in red and indicate with an arrow which way the bike went. What is the length of the bike frame?



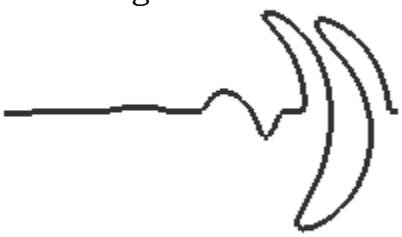
(Hint: *Note that there are four possible combinations depending on which track is the front/back track and one of two directions. Find the answer by testing all possibilities! A unique identification should be possible when we see that the tangent in one direction from one of the tracks will intersect the other track consistently at the same distance.*)

What if these had been the tire tracks left in the sand? Can you tell which is the front and which the back tire? Can you say which way the bike went?





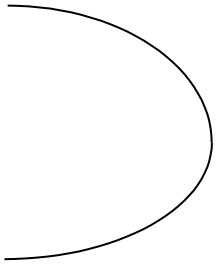
Imagine that Holmes and Watson discover a single tire track instead of a pair of tire tracks. First of all, could a bicycle have created this single tire track? And if so, could Holmes and Watson determine the direction the bank robber was heading?



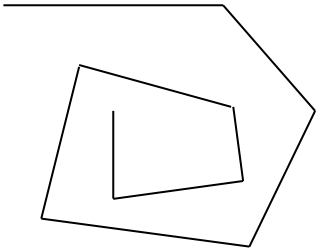
c) Recall from part a) that a bicyclix with a right angle turn has area a quarter of circle with radius equal to the length of the bicycle L (distance between wheels). In general, the area of any bicyclix is that of the circular sector with central angle equal to the angle of turn of the bicycle.

Note that if curve bends monotonously in the same direction, the swept-out area only depends on (the length and) the angle between the starting tangent vector and the last. Without drawing the actual sweep, use this observation to calculate the area of the tangent sweeps with tangent segments of length 1.

Area of the tangent sweep

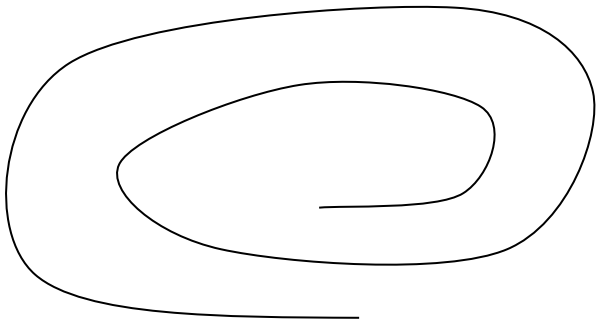


= _____.



Area of the tangent sweep

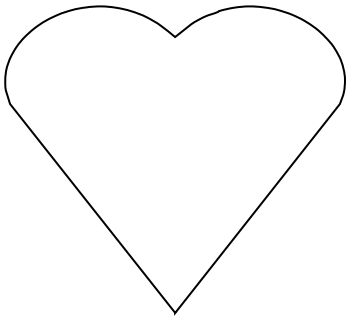
= _____.



Area of the tangent sweep

= _____.

Challenge: Is it easy to find the area of the tangent sweep for the following figure? Why or why not?



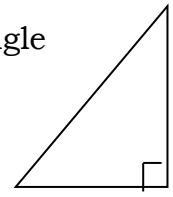
4. Area of Triangles

Remember the area formula for a triangle - it is equal to ½ times the height times the base of the triangle.

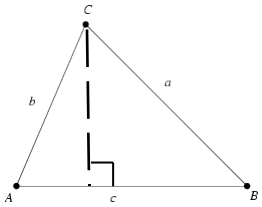
A = ½ * b * h

Before we do a few exercises with cutting and pasting areas to convince ourselves why it is always true, let's review some terminology:

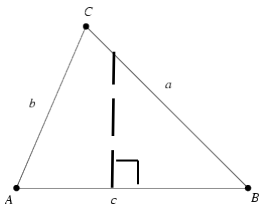
right triangle = triangle with one 90 degree angle



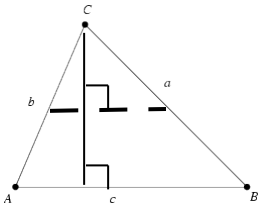
height = length of the line perpendicular to the base c and going through the opposite point C



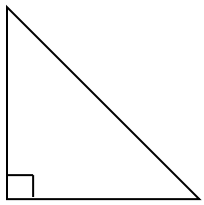
perpendicular bisector of the base = line through the midpoint between A and B and perpendicular to the base c



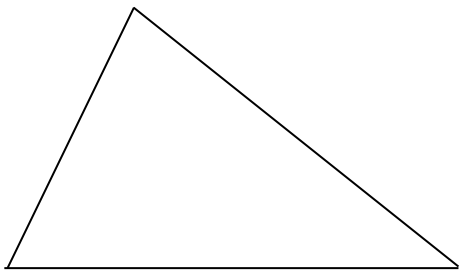
perpendicular bisector of the height = line through the midpoint of the height and perpendicular to it = line parallel to the base at half the height



a) Why is it easy to believe the area formula for a right triangle?

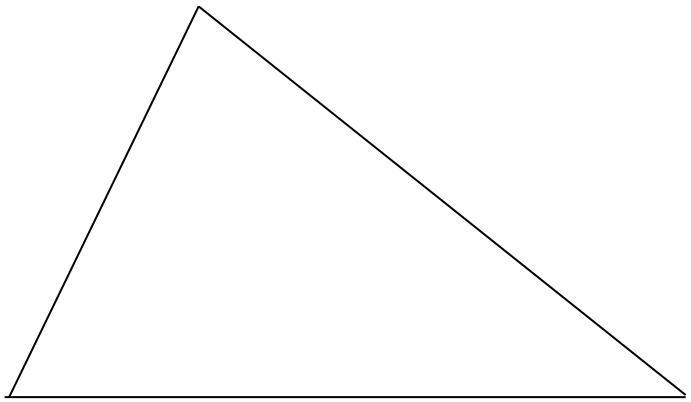


b) Transform the given triangle into a rectangle of the same area by cutting up and putting back together.



(Hint: *Draw the height and the perpendicular bisector of the height.*)

c) Show that any triangle can be transformed into right triangle of the same area by just cutting, sliding and possibly rotating pieces. (Find more triangle pictures at the end of the handout!)

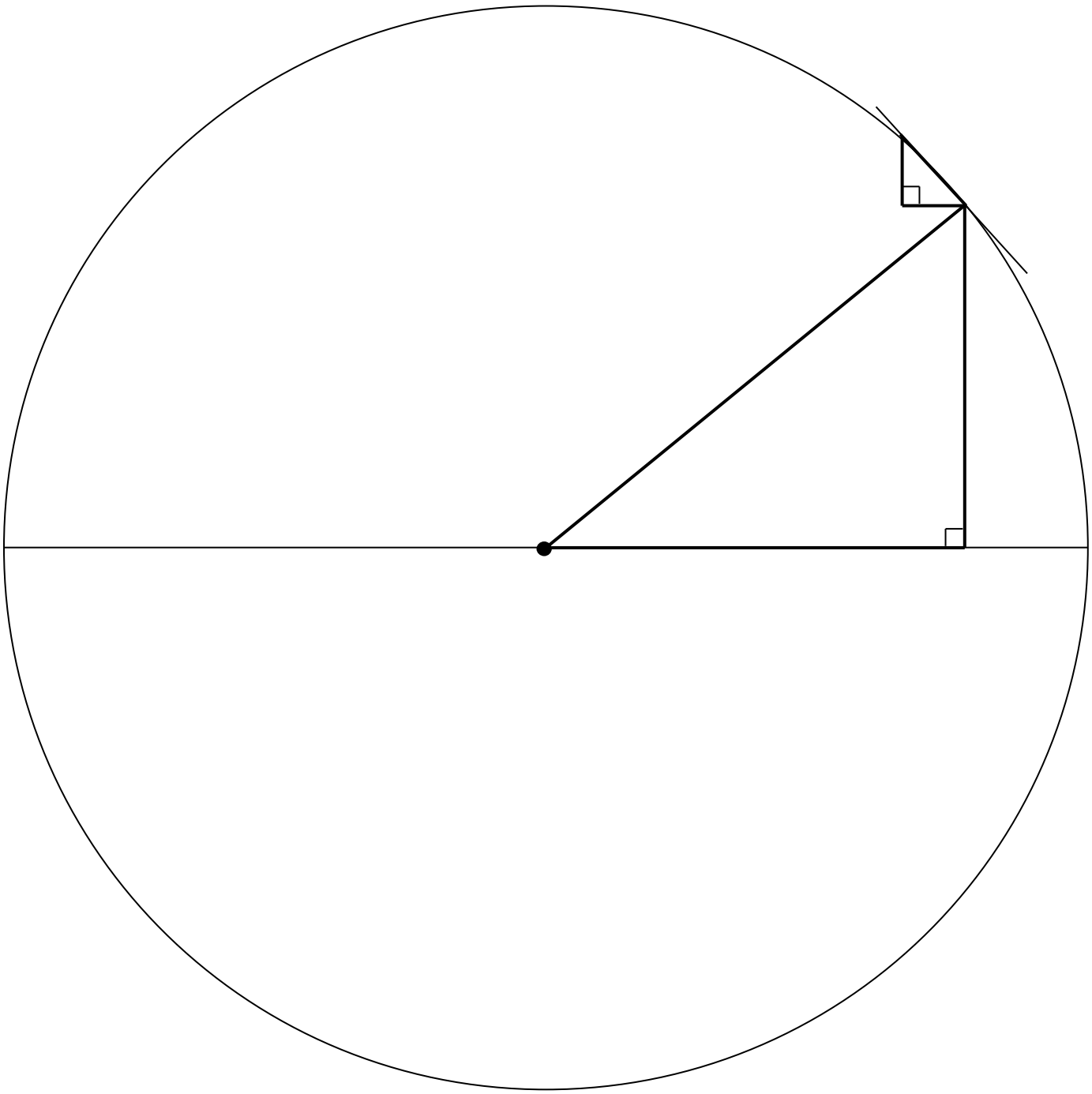


(Hint: *Draw the height and both the perpendicular bisector of the height as well as the perpendicular bisector of the base. Feel free to cut along those lines – maybe not all of them – and slide pieces around to get your result.)*

5. **Going Up in Dimension**

The surface area of a sphere of radius r is $A = 4\pi r^2$. Can you give a reason why this formula is true? By the end of this presentation you will!

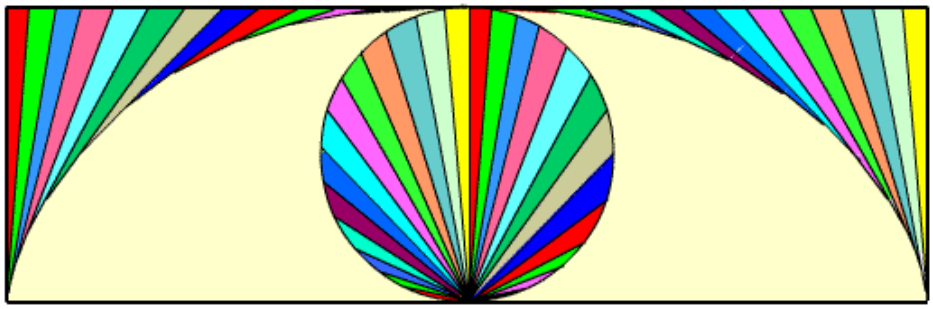
For the argument, it is going to be useful to know that the given triangles below are similar, and we will guide you through the proof of this.



6. **Take Home Challenges**

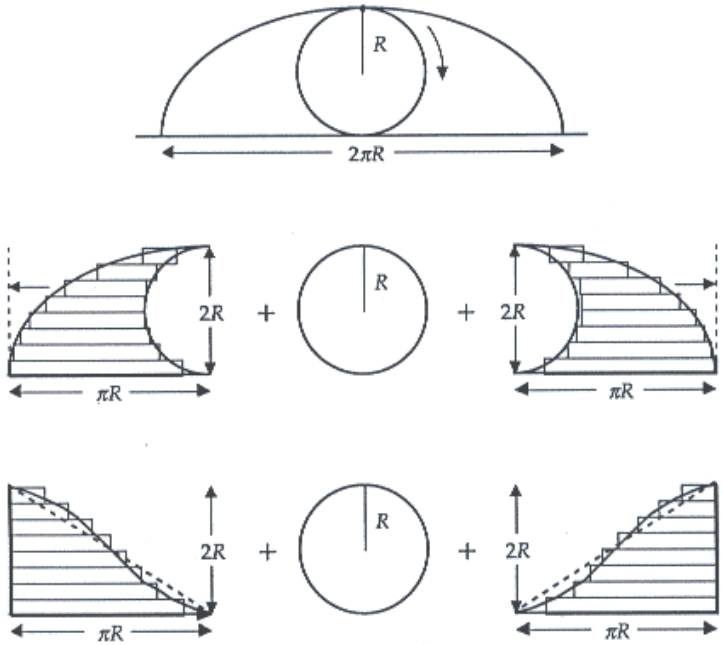
a) Area of a Cycloid
Here is another standard calculus problem – the cycloid, that is the curve traced out by a point on the perimeter of a circular disk that rolls along a horizontal line.

- Without deriving and equation for the cycloid or using calculus, use the suggestive picture below to try and show what the area of the region between the arch of the cycloid and the horizontal line is.



(Hint: *If the disk has diameter d = 2R, what is the area of the whole rectangle ?* Note that in this example, we again cut up and rearrange areas, only that this time the wedges do not have the same radius!)

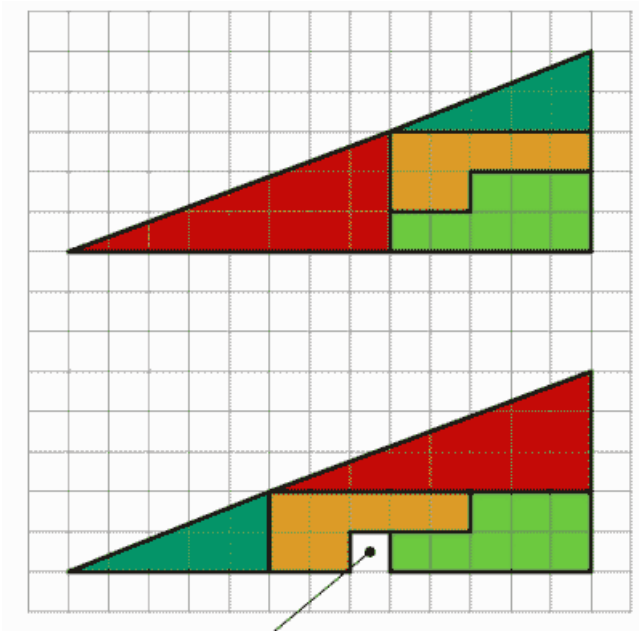
- Here is yet another way of computing the area under an arch of the cycloid by cutting and sliding – a proof without words discovered by Richard A. Beekman:



$$\frac{1}{2}\pi R \cdot 2R \quad + \quad \pi R^2 \quad + \quad \frac{1}{2}\pi R \cdot 2R$$

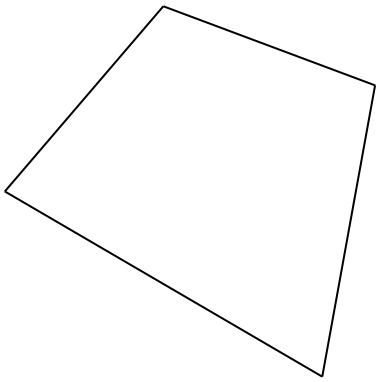
$$\Rightarrow A = \ ?$$

b) Another Mystery: From where comes the “hole”?

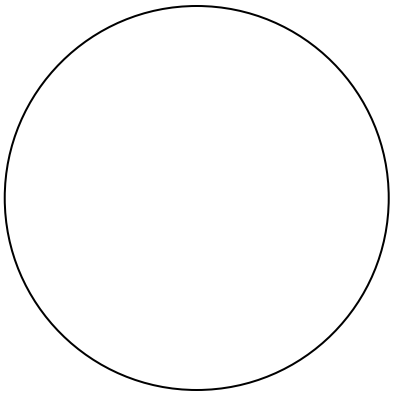


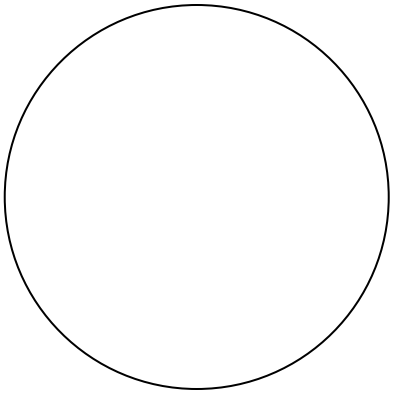
In the picture below, the shaded parts of the triangle have been moved around. The partitions are exactly the same as those used above – but from where comes the “hole”?

Drawing for #1 a)

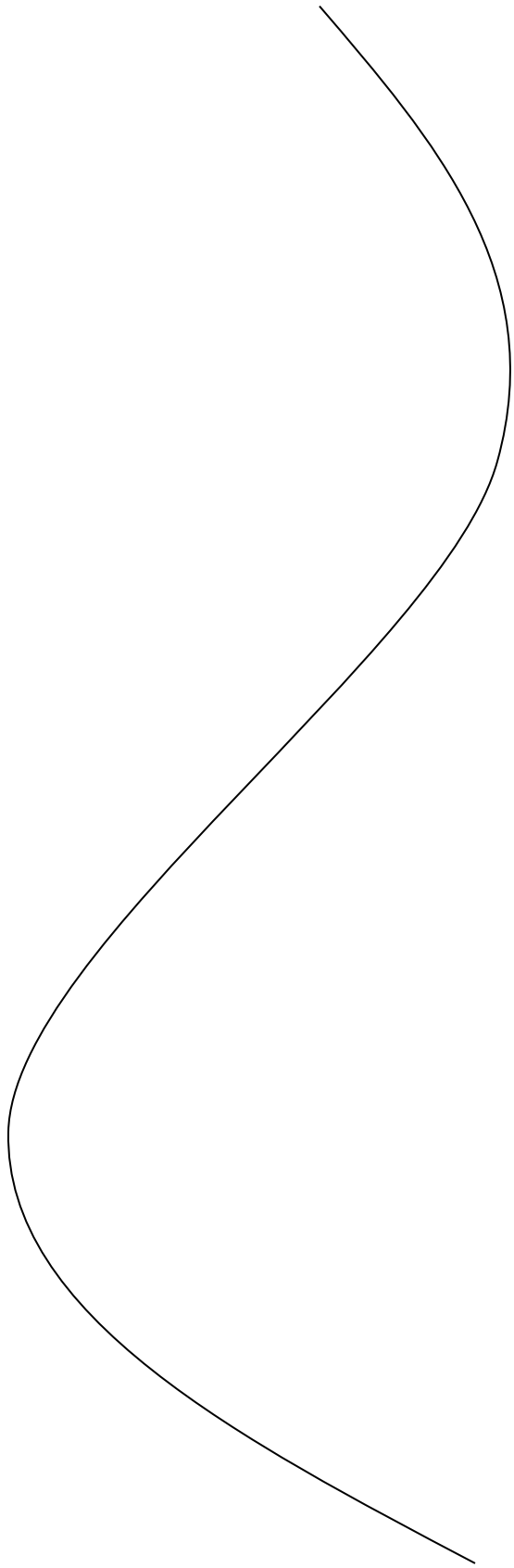


Drawing for #1 b)

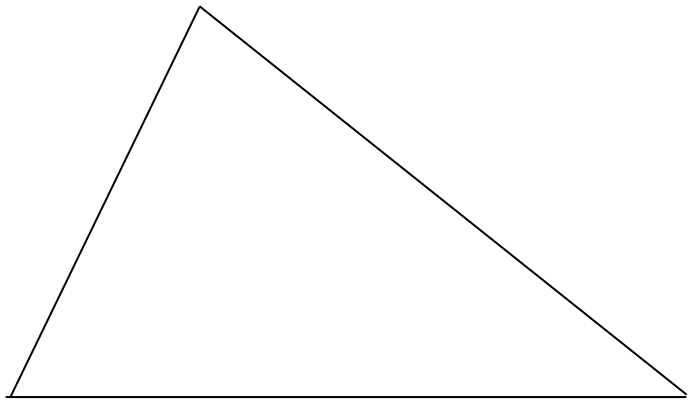




Drawing for #2 b)



Drawing for #4 c)



Here is a second triangle in case you need another try at this ☺