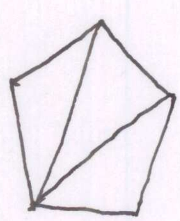
EASY AS 1, 2, 3 . . .

. . . or is it?

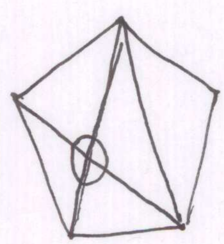
Two problems to think about:

1. Given a convex polygon with n sides, how many diagonals does the polygon have?

2. Given a convex polygon with n sides, how many ways are there to dissect the polygon into triangles using <u>non-intersecting</u> diagonals?



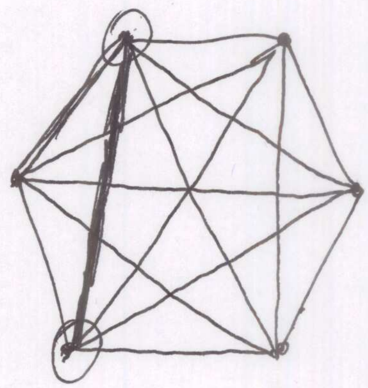
Legal



Illegal

COMBINATORICS

1. Counting Stuff
2. Understanding Finite Structures

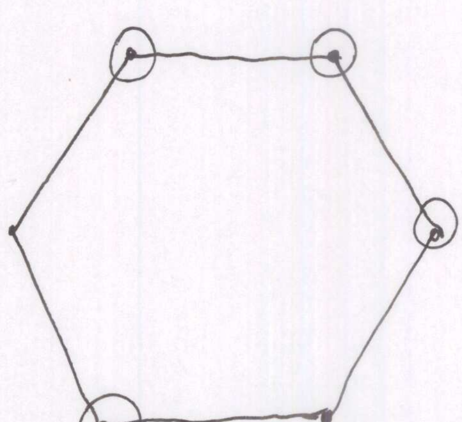


6-gon

$\rightarrow$ 9 diagonals.

How many diagonals does a 20-gon have?

How many diagonals does an n-gon have? (in terms of n)



<u>diagonal</u>

Connects two vertices.

(But not a side.)

6 vertices, and for each vertex
3 non-adjacent vertices

$$\frac{(6 \text{ vertices}) \times (3 \text{ non-adjacent vertices})}{} = \frac{18}{2}$$

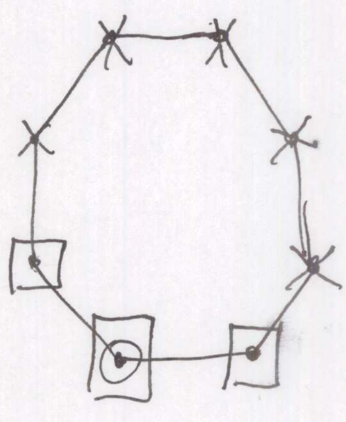
Choice #1   Choice #2

$$= 9$$

How many diagonals does an n-gon have?

Possible start points: n $\frac{n \cdot 3}{2}$

Possible end points: $\cancel{n/2}$ $n-3$



$n=8$

5 endpts.

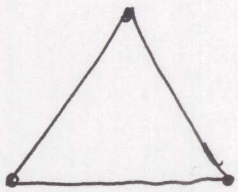
$n=6$

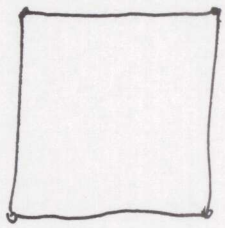
3 endpts.

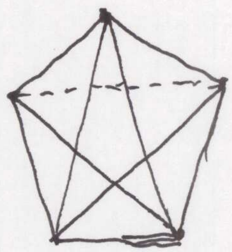
Diagonals: $\frac{n(n-3)}{2}$

Diagonals of a 20-gon: $\frac{20 \cdot 17}{2} = 170$

n (pic) diagonals

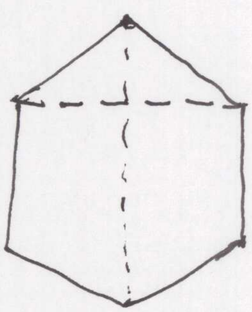
3  0 diagonals

4  2 diagonals

5 

Diagonals of the quadrilateral : 2
One side of the triangle : 1
From the top vertex : 2

$\overline{}$ 5

6 

Diagonals of the pentagon : 5
One side of the Δ : 1
From the top : 3

$\overline{}$ 9

Let D_n = # diagonals of an n-gon

$$D_n = D_{n-1} + 1 + (n-3)$$

$$= D_{n-1} + n - 2$$ Recursive formula

How many ways are there to make change for 30¢ using U.S. coins?

Q	D	N	P	
~~0~~	~~0~~	~~0~~	~~30~~	
1	0	1	0	2
1	0	0	5	
0	3	0	0	1
0	2	2	0	
0	2	1	5	3
0	2	0	10	
0	1	4	0	
0	1	3		
0	1	2		5
0	1	1		
0	1	0		
0	0	6		
		5		
		4		7
		3		
		2		
		1		
0	0	0	30	

18

QDNP(30) = DNP(30) + DNP(5)

The Easiest Change-Making Problem

ways to make change for 1¢

OR

ways to make change for n¢ 1

using pennies only.

ways using nickels and/or pennies

30¢

N	P
6	0
5	-
4	
3	
2	
1	
0	

7

ways using DNP

$$QDNP(100) = DNP(0) + DNP(25) + DNP(50) + DNP(75) + DNP(100)$$

Amt.	QDNP	DNP	NP	P
0	1	$\boxed{1}$	1	1
5	2	2 $1\cdot 2$	2	1
10	4	$\boxed{4}$	3	1
15	6	6 $2\cdot 3$	4	1
20	9	$\boxed{9}$	5	1
25	13	12 $3\cdot 4$	6	1
30	18	$\boxed{16}$	7	1

\# ways to make change for \$10.00 using DNP?

$= DNP(1000)$

$$NP(5) = P(5) + NP(0)$$

$$DNP(10) = \underbrace{NP(10)}_{3} + \underbrace{DNP(0)}_{1}$$

$$DNP(15) = NP(15) + DNP(5)$$

$DNP(0) = 1$

$DNP(\underline{1}0) = \underline{4}$

$DNP(\underline{2}0) = \underline{9} \qquad (2+1)^2$

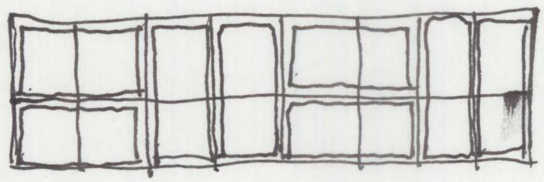
$\vdots$

$DNP(100\,0) = 101^2$

$= 10201$

New Question

How many ways are there to tile a $2 \times n$ floor using 2×1 tiles?



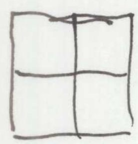
8

5

3

n	# Tilings of $2 \times n$
0	1
1	1
2	2
3	3
4	5
5	8

Fibonacci?!



How can we reduce a tiling problem to an easier problem?

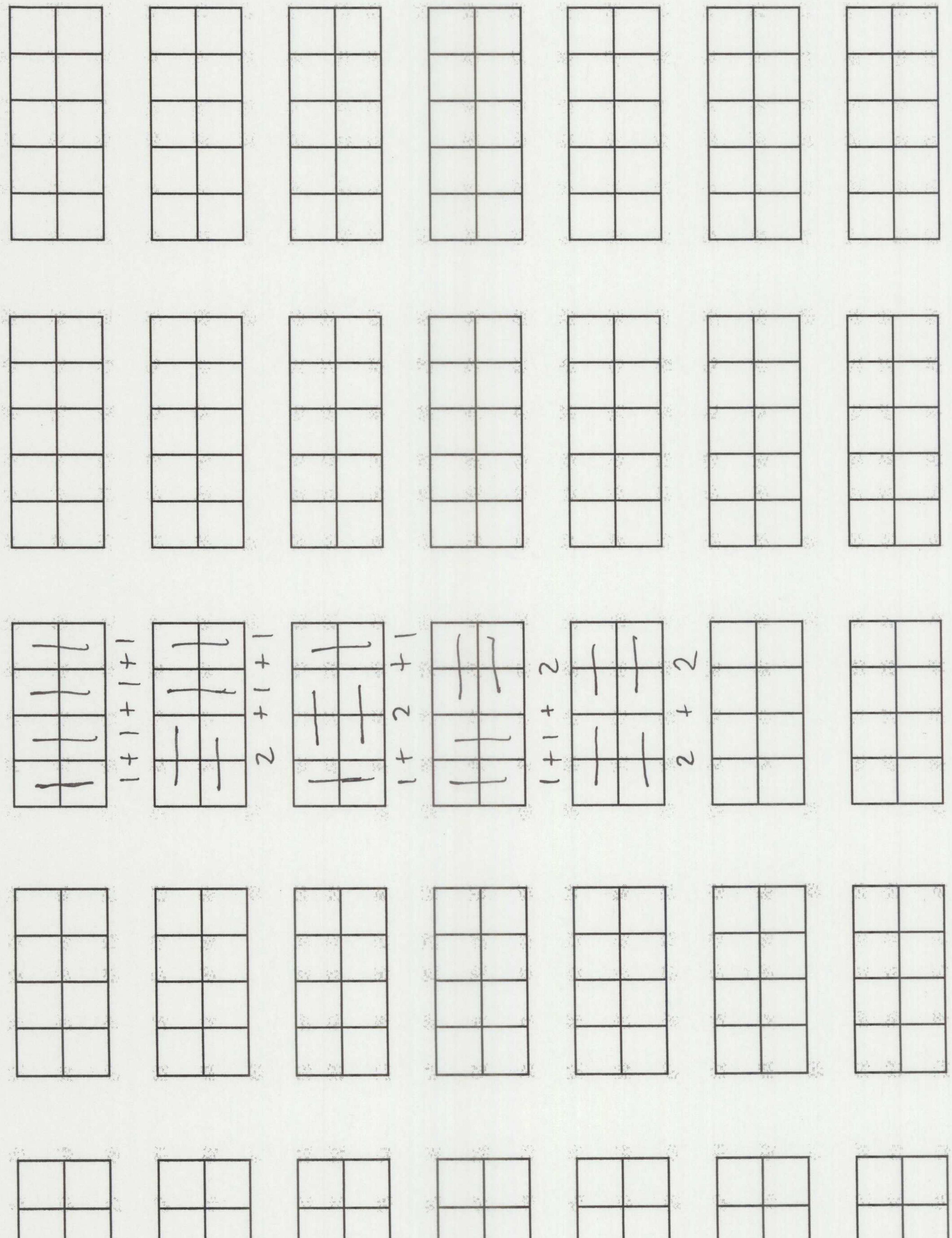
$$4 = 1 + \underbrace{\qquad\qquad}_{\text{need 3 to finish}}$$

$$4 = 2 + \underbrace{\qquad\qquad}_{\text{need 2 to finish}}$$

S_n = # ways to write n as a sum of 1's and 2's

$$S_4 = S_3 + S_2$$

$$S_n = S_{n-1} + S_{n-2}$$


1+1+1+1
2+1+1
1+2+1
1+1+2
2+2

F_n = # ways to tile a $2 \times n$ floor

$F_4 = F_3 + F_2$

5 3 2

$F_n = F_{n-1} + F_{n-2}$

Fibonacci Recursion

How many ways are there ~~to~~ to write n as a sum of 1's and 2's?

(Order Matters.)

$4 = 2 + 2$

$4 = 1 + 1 + 1 + 1$

$4 = 2 + 1 + 1$

$4 = 1 + 2 + 1$

$4 = 1 + 1 + 2$

5

Theorem Binet's Formula

ϕ = Golden Ratio $\qquad -\frac{1}{\phi}$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Generating Functions - Use an infinite series + ideas from calculus to solve a recurrence.