Moving Through Space with Geometric Algebras
Part II: Reflections and Rotations

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**INTRODUCTION**

**GEOMETRIC ALGEBRA**
  Scalars plus vectors plus bivectors!
  The vector product

**VECTOR DIVISION!**
  The inverse of a vector
  Vector rejection

**REFLECTION AND ROTATION**
  Projection minus rejection!
  Composing reflections

**CONCLUSION**
  Final remarks
What are the elements in the geometric algebra?

- Scalars: $\alpha, \beta$
- Vectors: $\mathbf{a}, \mathbf{b}$
- Bivectors $\mathbf{A}, \mathbf{B}$
- ...and sums of those! $\mathbf{a} = \alpha + \mathbf{a} + \mathbf{A}$
- To draw a proper picture of this, we would need eight dimensions!
The geometric algebra product is a combination of the outer and inner products

\[ ab = a \cdot b + a \land b \]

The product of two vectors is a scalar plus a bivector!

Many familiar properties:

- \( a(bc) = (ab)c \)
- \( a(b + c) = ab + ac \)
- \( (a + b)c = ac + bc \)
- \( \alpha a = a\alpha \) for scalars \( \alpha \)...

BUT \( ab \neq ba \)
0 AND 1

- There is a 0 element: $0 + a = a = a + 0$ for all $a$
- Subtraction: if $a = \alpha + \mathbf{a} + \mathbf{A}$, then $-a = -\alpha - \mathbf{a} - \mathbf{A}$, so $a - a = 0$
- There is a 1 element: $1(a) = a = (a)1$
- So we have addition, subtraction, multiplication, ...
Vector inversion: \( \mathbf{a}^{-1} \)

- Given \( \mathbf{a} \), we want a vector \( \mathbf{a}^{-1} \) such that \( \mathbf{a}\mathbf{a}^{-1} = 1 = \mathbf{a}^{-1}\mathbf{a} \)
- Recall: \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \cos \theta = |\mathbf{a}|^2 \)
- So define

\[
\mathbf{a}^{-1} = \frac{\mathbf{a}}{|\mathbf{a}|^2}
\]

Figure: The length of \( \mathbf{a}^{-1} \) is \( \frac{1}{|\mathbf{a}|^2} \)
EXERCISE SET 3

- If $a^{-1} = a/|a|^2$ is the inverse of $a$, then what is the inverse of $2a$ (in terms of $a^{-1}$)?
- Check that our definition for $a^{-1}$ is the right one by verifying $aa^{-1} = 1$.
- Challenge: Consider the silly equation $x = x(aa^{-1}) = (xa)a^{-1}$. Expand this using the product $xa$, the distributive property, and our definition for $a^{-1}$. What two terms do you get?? Hint: draw a picture.
Let’s look at that last exercise:

\[ x = x(aa^{-1}) \]
\[ = (xa)a^{-1} \]
\[ = (x \cdot a + x \wedge a)a^{-1} \]
\[ = (x \cdot a)a^{-1} + (x \wedge a)a^{-1} \]

But \((x \cdot a)a^{-1}\) is projection!

\[ (x \cdot a)a^{-1} = \frac{x \cdot a}{|a|^2}a \]
Vector rejection!

- So the rejection is:

$$\frac{x \wedge a}{|a|^2} a = x - \frac{x \cdot a}{|a|^2} a$$

Figure: A vector is its projection plus its rejection.
**Projection minus rejection!**

- Fact: \( axa^{-1} = (x \cdot a)a^{-1} - (x \wedge a)a^{-1} \)
- What is projection *minus* rejection?
Two reflections is a rotation!

- Assume $a$, $b$ are unit vectors: $|a| = 1$, $|b| = 1$
- Then $a^{-1} = a$ and $b^{-1} = b$
- Reflect twice: $x \mapsto bxb \mapsto a(bxb)a = abxba$
Use the fact that $\alpha a = a\alpha$ to show that $axa^{-1} = (x \cdot a)a^{-1} - (x \wedge a)a^{-1}$

Draw a picture representing the rotation $x \mapsto abxba$ in 3D.

Challenge: we’ve been rotating using vectors $a$ and $b$. What would it mean to rotate using bivectors $A$ and $B$? Can you draw a picture to represent $x \mapsto ABxBA$?
Geometric algebra is a powerful theoretical tool for Euclidean motion.

- Generalizes to other motions: translations, rotations about arbitrary lines.
- Although the ideas were discovered in the mid-1800s, scientists and engineers are only now starting to appreciate its full power...
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Main reference: *Geometric Algebra for Computer Science* by Dorst et al.

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