Arbitrage Pricing
What is an Equivalent Martingale Measure, and why should a bookie care?

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Introduction

What is Mathematical Finance?
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- Arbitrage Pricing Theory (APT)
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- Utility Maximization (and duality)
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- General Equilibrium Pricing
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- Arbitrage Pricing Theory (APT)
- Utility Maximization (and duality)
- General Equilibrium Pricing

We are going to focus on Arbitrage Pricing Theory.
What is Arbitrage?

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Definition

Arbitrage

- "Something for nothing"
- A chance to make money with no possibility of loss
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- "Something for nothing"
- A chance to make money with no possibility of loss

Why do I care?
What is Arbitrage?

**Definition**
Arbitrage

- "Something for nothing"
- A chance to make money with no possibility of loss

**Why do I care?**

- People move in to take advantage and it goes away.
- If I am a bank, I don’t want the prices of the instruments that I sell to allow arbitrage.
- Academics and practitioners may disagree.
An Example of No-Arbitrage Pricing

Suppose that we all know a bookie who will let us place an even money bet on the Butler / K. State game. As we will be considering a number of bets, let's fix a nice notation:

<table>
<thead>
<tr>
<th></th>
<th>$1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butler</td>
<td></td>
</tr>
<tr>
<td>K. State</td>
<td>$1</td>
</tr>
</tbody>
</table>

is the bet on the game that pays $1 if the Butler win, and costs us $1 if K. State wins.
An Example of No-Arbitrage Pricing

Similarly, if we bet two dollars on Butler to win, the bet is:

<table>
<thead>
<tr>
<th>Butler</th>
<th>$2</th>
</tr>
</thead>
<tbody>
<tr>
<td>K. State</td>
<td>$-2</td>
</tr>
</tbody>
</table>

And, if we bet a dollar that K. State will win, the bet is:

<table>
<thead>
<tr>
<th>Butler</th>
<th>$-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>K. State</td>
<td>$1</td>
</tr>
</tbody>
</table>
An Example of No-Arbitrage Pricing

In fact, we will assume that we can make any bet of the form:

\[ x \cdot \begin{pmatrix} \text{Butler} & $1 \\ -$1 & \end{pmatrix} = \begin{pmatrix} \text{Butler} & $x \\ -$x & \end{pmatrix} \]

for any \( x \) with our bookie. We will treat our bets as vectors which we can multiply by scalars with the obvious interpretation, and identify a constant payoff with the cash amount. That is

\[ \begin{pmatrix} \text{Butler} & $1 \\ $1 & \end{pmatrix} = $1. \]
An Example of No-Arbitrage Pricing

Now I ask you, how much is the payoff worth if I can borrow money without interest until after the game?

<table>
<thead>
<tr>
<th>Butler</th>
<th>$4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2</td>
</tr>
</tbody>
</table>
An Example of No-Arbitrage Pricing

Now I ask you, how much is the payoff worth if I can borrow money without interest until after the game?

<table>
<thead>
<tr>
<th>Butler</th>
<th>$4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kansas</td>
<td>$2</td>
</tr>
</tbody>
</table>

Answer: Exactly $3. Any other price admits arbitrage.
An Example of No-Arbitrage Pricing

Suppose that I can buy this payoff from some sucker for less than $3, say $2. I borrow $2, and buy the payoff. Then I bet $1 on the K. State with the bookie. After the game I have

\[
\begin{array}{c|c}
\text{Butler} & \$4 \\
\text{K. State} & \$2 \\
\end{array} + \begin{array}{c|c}
\text{Butler} & -\$1 \\
\text{K. State} & \$1 \\
\end{array} = \$3.
\]

I pay back the $2 I owe, and I put the free dollar in my pocket.
An Example of No-Arbitrage Pricing

Suppose that I can buy this payoff from some sucker for less than $3, say $2. I borrow $2, and buy the payoff. Then I bet $1 on the K. State with the bookie. After the game I have

\[
\begin{array}{c|c|c|c}
\text{Butler} & $4 & \text{Butler} & -$1 \\
\text{K-State} & $2 & \text{K-State} & $1 \\
\end{array}
\]

\[= $3.\]

I pay back the $2 I owe, and I put the free dollar in my pocket.

In fact, I do this as much as I possibly can, until I can’t find anyone willing to sell me the payoff for less than $3.
An Example of No-Arbitrage Pricing

Conversely, suppose that I can sell this payoff to some sucker for more than $3, say $4. I sell the payoff for $4, and bet $1 on Butler. After the game I have

\[
\begin{array}{c|c|c}
\text{Butler} & $1 & \text{Butler} & $5 \\
\hline
\text{Butler} & -$1 & & $3 \\
\end{array}
\]

$4 + \begin{array}{c|c}
\text{Butler} & $1 \\
\hline
\text{Butler} & -$1 \\
\end{array} = \begin{array}{c|c}
\text{Butler} & $5 \\
\hline
\text{Butler} & $3 \\
\end{array}$
An Example of No-Arbitrage Pricing

Conversely, suppose that I can sell this payoff to some sucker for more than $3, say $4. I sell the payoff for $4, and bet $1 on Butler. After the game I have

\[
\begin{array}{c}
4 \\
\downarrow \\
\text{Butler} \\
1 \\
\hline \\
-1 \\
\end{array} \quad = \quad \begin{array}{c}
5 \\
\downarrow \\
\text{Butler} \\
3 \\
\end{array}
\]

Of course, I now have to make due on the payoff that I sold, so I really have

\[
\begin{array}{c}
5 \\
\downarrow \\
\text{Butler} \\
3 \\
\hline \\
4 \\
\downarrow \\
\text{Butler} \\
2 \\
\end{array} \quad = \quad 1.
\]

I then put the free dollar in my pocket.
No-Arbitrage Pricing

This is the essence of No-Arbitrage pricing.

It's not about gambling, or making money in the long term.

Instead, the bets that the bookie offers effectively set the prices for all possible payoffs.

If we do not respect these prices, people can make free money off of us without taking any risk.
The Equivalent Martingale Measure

How do I find arbitrage?

How do I ensure that the instruments that I sell do not admit arbitrage?
The Equivalent Martingale Measure

How do I find arbitrage?

How do I ensure that the instruments that I sell do not admit arbitrage?

“Equivalent Martingale (Probability) Measure” (EMM) or “Risk-Neutral Martingale (Probability) Measure”

This is the central tool of arbitrage pricing theory.
Recall the Classical Probability Theory Setup

\( \Omega \) is the set of possible “events.”
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\( \mathbb{P} \) assigns “probabilities” in \([0, 1]\) to subsets of \( \Omega \) with \( \mathbb{P}(\Omega) = 1 \). We think of these numbers as “relative likelihood,” with 1 being certainty.

\( \mathbb{P}(\text{The sun will rise tomorrow}) = 0.999999 \)
Recall the Classical Probability Theory Setup

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A random variable, \( X \), is a function from \( \Omega \) to \( \mathbb{R} \). We read this as, “if event \( \omega \in \Omega \) happens, then \( X \) has value \( X(\omega) \)”
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A random variable, \( X \), is a function from \( \Omega \) to \( \mathbb{R} \). We read this as, “if event \( \omega \in \Omega \) happens, then \( X \) has value \( X(\omega) \)”

In our setup, \( \Omega = \{\) Petro, TCU\( \} \) and a random variable is a payoff. We don’t know \( \mathbb{P} \) yet.
The Equivalent Martingale Measure

We want to find $\mathbb{P}$ so that all the bets the bookie offers us are “fair.” In fact, “martingale” is mathematical jargon that essentially means fair game.

By fair, we mean that they have expectation 0.

Recall that expectation is given by

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} \mathbb{P}(\omega) \cdot X(\omega)$$

We read this as averaging together all the possible values of $X$, and weighting those values by their likelihood.
The Equivalent Martingale Measure

In fact, since all the bets are just multiples of $1 - 1$,
if we find $\mathbb{P}$ such that the bet on the Butler is "fair", then all bets are "fair", as

\[
\mathbb{E}[xX] = x \mathbb{E}[X]
\]
The Equivalent Martingale Measure

In fact, since all the bets are just multiples of

<table>
<thead>
<tr>
<th>Butler</th>
<th>$1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-Butler</td>
<td>-$1</td>
</tr>
</tbody>
</table>

if we find \( \mathbb{P} \) such that the bet on the Butler is "fair", then all bets are "fair", as

\[
\mathbb{E}[xX] = x \mathbb{E}[X]
\]

Later we will also use that

\[
\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]
\]

Expectation is (just like) integration.
The Equivalent Martingale Measure

If the bet on the Butler is "fair", we must have

\[
0 = \mathbb{E} \left[ \begin{array}{cc}
\text{Butler} & $1 \\
\text{Butler} & -$1
\end{array} \right] = P(\text{Butler}) \cdot 1 + P(\text{KState}) \cdot (-$1)
\]

and \( P(\text{Butler}) + P(\text{KState}) = 1 \).
In fact we also require that $P(\text{Butler}) > 0$, $P(\text{K-State}) > 0$.

This is the “equivalent” part. Equivalent is mathematical jargon which means that the same events which can happen in the real world can happen in a world ruled by the equivalent martingale measure.
The only solution to this system is

\[
\mathbb{P}(\text{Butler}) = \mathbb{P}(\text{K-State}) = \frac{1}{2}
\]

This is the Equivalent Martingale Measure, and we can use it to compute the no arbitrage price of a payoff.

This is much easier than trying to give an arbitrage argument as we did in the example above.
Revisiting the Example

To compute the no arbitrage price of a payoff, we just compute its expectation under the EMM. For example,

\[
\mathbb{E} \begin{bmatrix}
\text{Butler} & \$4 \\
\text{K"{a}rek} & \$2 \\
\end{bmatrix}
= \mathbb{P}(\text{Butler}) \cdot \$4 + \mathbb{P}(\text{K"{a}rek}) \cdot \$2 \\
= \frac{1}{2} \cdot \$4 + \frac{1}{2} \cdot \$2 \\
= \$3
\]

Which agrees with the odd-hock arguments we made above.
In fact, the payoff $4-2$ is really just $3$ with a $1$ bet on the Butler mixed in.

Since a bet on the K. State cancels out a bet on the Butler, it’s now clear why this payoff must be worth exactly $3$. 
Revisiting the Example

Moreover, since all bets are "fair" under the EMM, we can see why the expectation operator gives us the risk-free, cash equivalence for the payoff:

\[
\begin{bmatrix}
$4 \\
$2
\end{bmatrix} = \begin{bmatrix}
$3 \\
\end{bmatrix} + \begin{bmatrix}
$1 \\
-$1
\end{bmatrix}
\]

\[
= \mathbb{E}[3] + \mathbb{E}
\]

\[
= 3 + 0
\]
The EMM is not the "true" probability of an event happening.

We don’t claim to know the "true" probability of the event happening. People who think Butler are better than K. State can make that bet, and we cannot arbitrage them.

The EMM just expresses the probabilities that are somehow imbedded in the bets that the bookie offers us. It allows us to see all the consequences of the odds that the bookie has set.
Suppose now that the bookie will also offer us even money bets on the WVU / Kentucky game. Now we have

$$\Omega = \left\{ \begin{array}{c}
\text{WVU, } \text{UK, } \text{Butler, } \text{WVU} \\
\text{Butler, } \text{WVU, } \text{UK} \\
\text{WVU, } \text{UK, } \text{Butler} \\
\text{UK, } \text{WVU, } \text{Butler}
\end{array} \right\}.$$ 

is the event that Butler and WVU win.

is the event that Butler and Kentucky win.

is the event that K. State and WVU win.

is the event that K. State and Kentucky win.
We will now write payoffs as

<table>
<thead>
<tr>
<th></th>
<th>WVU</th>
<th>UK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butler</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>KU</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

where the row indicates the winner of the first and the column indicates the winner of the second game.
Some Examples

A $1 bet on the Butler to win the first game is

\[
\begin{array}{c|cc}
\text{Butler} & \text{WV} & \text{UK} \\
\hline
$1 & $1 \\
\hline
-\$1 & -\$1 \\
\end{array}
\]

A $5 bet on the Kentucky to win the second game is

\[
\begin{array}{c|cc}
\text{Butler} & \text{WV} & \text{UK} \\
\hline
-\$5 & $5 \\
\hline
-\$5 & $5 \\
\end{array}
\]
The General Form of a Bet

In general, a bet looks like a combination of bets on both games:

\[
\begin{pmatrix}
    \text{West Virginia} & \text{Kentucky} \\
    \text{Butler} & \text{West Virginia} \\
    \text{Kentucky} & \text{Butler}
\end{pmatrix}
\]

\[
x \begin{pmatrix}
    $1 \\
    -$1
\end{pmatrix}
+ y \begin{pmatrix}
    $1 \\
    -$1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    $(x + y)$ \\
    $-(x - y)$
\end{pmatrix}
\]

\[
\begin{pmatrix}
    $(x - y)$ \\
    $-(x + y)$
\end{pmatrix}
\]
An Equivalent Martingale Measure

Once again, we try to find an EMM. And, once again, its enough to check that the generating bets are “fair”, as

\[
\mathbb{E}[x X + y Y] = x \mathbb{E}[X] + y \mathbb{E}[Y] = x \cdot 0 + y \cdot 0 = 0
\]

if \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[Y] = 0 \). That is, if \( X \) and \( Y \) are "fair" bets.
An Equivalent Martingale Measure

We can easily check that \( \mathbb{P} \), given by,

\[
\begin{align*}
\mathbb{P} \left( \begin{array}{c}
\text{West Virginia} \\
\text{Butler}
\end{array} \right) &= \mathbb{P} \left( \begin{array}{c}
\text{Kentucky} \\
\text{Butler}
\end{array} \right) = \mathbb{P} \left( \begin{array}{c}
\text{West Virginia} \\
\text{West Virginia}
\end{array} \right) = \mathbb{P} \left( \begin{array}{c}
\text{Kentucky} \\
\text{Kentucky}
\end{array} \right) = \frac{1}{4}
\end{align*}
\]

is an EMM. Just see if the two generating bets are “fair”.
Check First Game Bet for $P$

\[
E \begin{bmatrix}
\text{WV} & \text{UK} \\
\text{Butler} & 1 & 1 \\
\text{KState} & -1 & -1 \\
\end{bmatrix} = P(\text{Butler}) \cdot 1 + P(\text{Butler}) \cdot 1 \\
- P(\text{WV}) \cdot 1 - P(\text{UK}) \cdot 1 \\
= \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 \\
- \frac{1}{4} \cdot 1 - \frac{1}{4} \cdot 1 \\
= 0
\]
Check Second Game Bet for $P$

\[
\begin{bmatrix}
\text{WV} & \text{UK} \\
\text{Butler} & $1$ & -$1$ \\
\text{K} & $1$ & -$1$
\end{bmatrix}
\]

\[
= P(\text{WV}) \cdot $1 - P(\text{UK}) \cdot $1
\]

\[
+ P(\text{WV}) \cdot $1 - P(\text{UK}) \cdot $1
\]

\[
= \frac{1}{4} \cdot $1 - \frac{1}{4} \cdot $1
\]

\[
+ \frac{1}{4} \cdot $1 - \frac{1}{4} \cdot $1
\]

\[
= $0
\]
Is this the only EMM?

As both bets are “fair” under \( P \), \( P \) is an EMM. Is it the only one?
Answer:

\[ \hat{P}(X) = \frac{1}{3} \hat{P}(X) = \frac{1}{6}. \]

This still sums to one, and we can check that the bets are still “fair”. We use \( \hat{E}[X] \) to denote the expectation of \( X \) under \( \hat{P} \).
Is this the only EMM?

As both bets are “fair” under $\mathbb{P}$, $\mathbb{P}$ is an EMM. Is it the only one?

Answer: No.

Consider $\hat{\mathbb{P}}$ given as:

$$
\hat{\mathbb{P}}\left(\begin{array}{c}
\text{Butler} \\
\text{West Virginia}
\end{array}\right) = \hat{\mathbb{P}}\left(\begin{array}{c}
\text{Butler} \\
\text{Kentucky}
\end{array}\right) = \frac{1}{3}
$$

$$
\hat{\mathbb{P}}\left(\begin{array}{c}
\text{Butler} \\
\text{Kentucky}
\end{array}\right) = \hat{\mathbb{P}}\left(\begin{array}{c}
\text{West Virginia} \\
\text{Kentucky}
\end{array}\right) = \frac{1}{6}.
$$

This still sums to one, and we can check that the bets are still “fair”. We use $\hat{\mathbb{E}}[X]$ to denote the expectation of $X$ under $\hat{\mathbb{P}}$. 
Check First Game Bet for $\hat{P}$

\[
\hat{E} = \begin{array}{cc}
\text{WVU} & \text{UK} \\
\text{Butler} & 1 & 1 \\
\text{K-State} & -1 & -1
\end{array}
\]

\[
\begin{align*}
\hat{P}(\text{Butler}) \cdot 1 + \hat{P}(\text{UK}) \cdot 1 - \hat{P}(\text{WVU}) \cdot 1 - \hat{P}(\text{K-State}) \cdot 1 \\
= \frac{1}{3} \cdot 1 + \frac{1}{6} \cdot 1 - \frac{1}{6} \cdot 1 - \frac{1}{3} \cdot 1 \\
= 0
\end{align*}
\]
Check Second Game Bet for $\hat{P}$

\[
\hat{E} = \begin{bmatrix}
WV & UK \\
BUTLER & $1$ & -$1$ \\
KState & $1$ & -$1$
\end{bmatrix}
= \hat{P}(WV) \cdot $1 - \hat{P}(BUTLER) \cdot $1
+ \hat{P}(WV) \cdot $1 - \hat{P}(UK) \cdot $1
\]

\[
= \frac{1}{3} \cdot $1 - \frac{1}{6} \cdot $1
+ \frac{1}{6} \cdot $1 - \frac{1}{3} \cdot $1
\]

\[
= $0
\]
In the first example, there was a two dimensional space of payoffs, and we had two generating payoffs: the constant payoff, and the bet. As a result, we could reproduce any payoff as a combination of the two generating payoffs.

Now we have a four dimensional space of payoffs, but we only have three generating payoffs: the constant payoff, and two bets. As a result, we can’t possibly replicate every possible payoff. When this happens we get a bunch of EMM’s.
Claims that We Can Replicate

Consider a payoff that we can replicate

\[
\begin{array}{c|c|c}
\text{WV} & \text{UK} \\
\hline
\text{Butler} & 6 & 0 \\
\hline
\text{KU} & 2 & -4 \\
\end{array}
\]

We know we can replicate this payoff because we can write it as:

\[
1 + 2 \left( \begin{array}{c|c|c}
\text{WV} & \text{UK} \\
\hline
\text{Butler} & 1 & 1 \\
\hline
\text{KU} & -1 & -1 \\
\end{array} \right) + 3 \left( \begin{array}{c|c|c}
\text{WV} & \text{UK} \\
\hline
\text{Butler} & 1 & -1 \\
\hline
\text{KU} & 1 & -1 \\
\end{array} \right)
\]
Take the Expectation of The Payoff Under \( P \)

\[
\begin{bmatrix}
\text{WV} & \text{UK} \\
\text{Butler} & 6 & 0 \\
\text{Kitts} & 2 & -4
\end{bmatrix}
= P(\text{Butler}) \cdot 6 + P(\text{UK}) \cdot 0 \\
+ P(\text{WV}) \cdot 2 - P(\text{UK}) \cdot 4
\]

\[
= \frac{1}{4} \cdot 6 + \frac{1}{4} \cdot 0 \\
+ \frac{1}{4} \cdot 2 - \frac{1}{4} \cdot 4
\]

\[
= 1
\]
Take the Expectation of The Payoff Under $\hat{P}$

$$\hat{E} = \begin{bmatrix}
\text{BUTLER} & \text{WV} & \text{UK} \\
\$6 & 0 \\
\$2 & -4 \\
\end{bmatrix} = \hat{P}(\text{BUTLER}) \cdot \$6 + \hat{P}(\text{BUTLER}) \cdot 0 \\
+ \hat{P}(\text{WV}) \cdot \$2 - \hat{P}(\text{UK}) \cdot 4$$

$$= \frac{1}{3} \cdot \$6 + \frac{1}{6} \cdot 0 \\
+ \frac{1}{6} \cdot \$2 - \frac{1}{3} \cdot 4$$

$$= \$1$$
All the EMM’s Must Agree on the Claims Which are Replicable

Of course, this is kind of obvious, as a replicable claim must look like

\[ c + xX + yY \]

where \( X \) is the bet on the first game, and \( Y \) is the bet on the second game. Then

\[
\tilde{E}[c + xX + yY] = \tilde{E}[c] + x\tilde{E}[X] + y\tilde{E}[Y] = c
\]

for any \( \tilde{P} \) which is an EMM, since \( X \) and \( Y \) must be “fair” bets under \( \tilde{P} \).
Completing the Market

Suppose now that our bookie will let us place an even money bet on the second game after we see the result of the first game. In effect, this breaks the bet on the second game:

\[
\begin{array}{c|cc}
& \text{WV} & \text{UK} \\
\hline
\text{Butler} & $1 & -$1 \\
\text{Kans} & $1 & -$1 \\
\end{array}
\]

into two different bets:

\[
\begin{array}{c|cc}
& \text{WV} & \text{UK} \\
\hline
\text{Butler} & $1 & -$1 \\
\text{Kans} & $0 & $0 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|cc}
& \text{WV} & \text{UK} \\
\hline
\text{Butler} & $0 & $0 \\
\text{Kans} & $1 & -$1 \\
\end{array}
\]
Is $\mathbb{P}$ still an EMM?

We know what we have to do to answer this question.

We have to check whether the two new bets that are available to us are “fair” under $\mathbb{P}$. 
Check First New Bet for $P$

\[
\begin{array}{c|cc}
\text{BET} & \text{WV} & \text{UK} \\
\hline
\text{BUTLER} & $1 & -$1 \\
\text{KANSAS} & $0 & $0 \\
\end{array}
\]

\[
E = P(\text{BUTLER}) \cdot $1 - P(\text{BUTLER}) \cdot $1 \\
+ P(\text{WV}) \cdot $0 + P(\text{UK}) \cdot $0
\]

\[
= \frac{1}{4} \cdot $1 - \frac{1}{4} \cdot $1 \\
+ \frac{1}{4} \cdot $0 + \frac{1}{4} \cdot $0
\]

\[
= $0
\]
Check Second New Bet for $P$

\[
\begin{bmatrix}
\text{WV} & \text{UK} \\
\text{Butler} & $0 & $0 \\
\text{Kansas} & $1 & -$1 \\
\end{bmatrix}
\]

\[
E = P(\text{WV}) \cdot $0 + P(\text{UK}) \cdot $0 + P(\text{Butler}) \cdot $1 - P(\text{UK}) \cdot $1
\]

\[
= \frac{1}{4} \cdot $0 + \frac{1}{4} \cdot $0 + \frac{1}{4} \cdot $1 - \frac{1}{4} \cdot $1
\]

\[
= $0
\]
Is $P$ still an EMM?

Answer:
Is $\mathbb{P}$ still an EMM?

Answer: Yes.

Both of the new bets are “fair” under $\mathbb{P}$, so $\mathbb{P}$ is still an EMM with respect to this strictly larger set of bets.
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How about $\hat{\mathbb{P}}$? Well, we check it now.
Check First New Bet for \( \hat{P} \)

\[
\hat{E} = \begin{bmatrix}
\text{Butler} & \text{WV} & \text{UK} \\
$1 & -$1 \\
$0 & $0 \\
\end{bmatrix}
\]

\[
= \hat{P}(\text{Butler}) \cdot $1 - \hat{P}(\text{Butler}) \cdot $1 \\
+ \hat{P}(\text{WV}) \cdot $0 + \hat{P}(\text{UK}) \cdot $0
\]

\[
= \frac{1}{3} \cdot $1 - \frac{1}{6} \cdot $1 \\
+ \frac{1}{6} \cdot $0 + \frac{1}{3} \cdot $0
\]

\[
= \frac{1}{6} \neq $0
\]
is No Longer an EMM

By adding more bets, we make it possible to replicate more claims. Since every EMM must assign the same expectation to every replicable claim, more claims means its harder to be an EMM.
\( \hat{P} \) is No Longer an EMM

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In fact, \( \hat{P} \) no longer works, and one can check that \( \hat{P} \) is now the only EMM.
P is No Longer an EMM

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So we see that how this works: more replicable payoffs \( \Leftrightarrow \) fewer EMMs.
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So we see that how this works: more replicable payoffs \( \Leftrightarrow \) fewer EMMs.

A claim is replicable iff there is only one EMM. When every claim is replicable, we say that the market is complete.
Conclusion

When we look at a system of bets, there are essentially three possibilities.
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- If there is an arbitrage possibility, then there is no EMM.
- If there are no arbitrage possibilities, then there is at least one EMM.
Conclusion

When we look at a system of bets, there are essentially three possibilities.

- If there is an arbitrage possibility, then there is no EMM.

- If there are no arbitrage possibilities, then there is at least one EMM.

- If every payoff is replicable, then there is exactly one EMM.