How Do You Group It?

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Mathematics Department
The University of Texas

October 24, 2010
Famous statements from science

• The Poincare Conjecture: Every closed 3-manifold with trivial fundamental is homeomorphic to the 3-sphere. - Grigori Perelman
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- Molecular symmetry: Molecules can be classified using groups.
- Crystal optics: The symmetry properties of a crystal are determined by its space group.
- Music Theory: The circle of fifths can be given the structure of a cyclic group.
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- Music Theory: The circle of fifths can be given the structure of a cyclic group.
What does “group” mean?

• (General) any collection of persons or things.
• (Chemistry) two or more atoms specifically arranged.
• (Music) a section of an orchestra.
• (Army) a flexible administrative and tactical unit

• “Group” has a specific mathematical definition, too.
So what IS a mathematical group?

Let’s investigate a group that we have all seen before:

The Integers!
Our observations about integers

• Adding two integers together gives you another integer.
• It does not matter where the parenthesis go—you still get the same answer.
• Adding zero to any number does nothing.
• Every integer has an inverse, like $7 + (-7) = 0$
A group $G$ is a set of objects (called elements), and an operation $\circ$ (called composition) that obeys four rules. For all group elements $a$, $b$, and $c$ we have:

**Closure:** $a \circ b$ is also in $G$.

**Associativity:** $(a \circ b) \circ c = a \circ (b \circ c)$

**Identity:** There's a special *identity* element $e$ such that

$$e \circ a = a \circ e = a$$

**Inverses:** Every element $a$ has an inverse $a^{-1}$ such that

$$a \circ a^{-1} = a^{-1} \circ a = e$$
Which of these are groups?

- Even integers with addition?

  A: Yes.

- Odd integers with addition?

  A: No.

- All the integers with multiplication?

  A: No.
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Ammonia

101.7 pm

107.8°
Rotations of an equilateral triangle

<table>
<thead>
<tr>
<th>Counter-clockwise rotation</th>
<th>Letters in order</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do nothing (0°)</td>
<td></td>
<td>$R_0$</td>
</tr>
<tr>
<td>Rotate once (120°)</td>
<td></td>
<td>$R_1$</td>
</tr>
<tr>
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<td></td>
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</tr>
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<td>$R_3$</td>
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1. If we rotate some more, what is going to happen with the letters?
2. Does $ACB$ ever show up? Why or why not?
The rotations themselves form a group. Does this make sense?

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The rotations make a group

Let’s check our understanding that the rotations are indeed a group.

1. How many distinct rotational symmetries does the triangle have?
2. How is this group different from the integers?
3. Inverses are pretty interesting in the rotation group.
   What is the inverse of \( R_0 \)?
   What is the inverse of \( R_1 \)?
   What is the inverse of \( R_2 \)?
Rotations and reflections

Now let’s consider a group made of both rotations and reflections. Let’s look at how reflections act on the letters:

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<th>Letters in order</th>
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the triangle VS the group

\[ R_2 \text{ followed by } S_0 \text{ is the same thing as } S_1: \]

\[ ABC \xrightarrow{R_2} BCA \xrightarrow{S_0} BAC \]

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the triangle VS the group

$R_2$ followed by $S_0$ is the same thing as $S_1$:

$$ABC \xrightarrow{R_2} BCA \xrightarrow{S_0} BAC$$

$$ABC \xrightarrow{S_1} BAC$$

We write:

$$S_0 \circ R_2 = S_1$$
The Cayley Table

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An important observation:

- This group is not Abelian.
The Dihedral Group $D_n$

This group is called $D_n$, the Dihedral Group.

- The elements of $D_n$ are rotations and reflections of a regular polygon with $n$ sides.
- There are $2n$ elements total.
- $D_n$ is finite.
- $D_n$ is non-Abelian (for $n \geq 3$).
- $D_n$ can be described by

$$\langle R, S \mid R^n = 1, S^2 = 1, SRS = R^{-1} \rangle$$
The Platonic Solids - regular convex polygons

- Cube
- Tetrahedron
- Octahedron
- Dodecahedron
- Icosahedron
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Kepler’s Model of the Solar System
IN THE NEWS

Matter & Energy

Elusive symmetry appears in nature
Complex E₈ patterns detected in supercold physical system

By Rachel Ehrenberg

A beautiful math emerges from the acrobatic flips of supercold atoms in a magnetic field.

In the Jan. 8 Science, researchers report detecting an elusive, complex symmetry known as the E₈ Lie group, long analyzed on paper but never before seen in a physical system. The work suggests that this numerical grace may be hidden in other systems and may provide a mathematical link between quantum processes in matter and the physics of the cosmos.

“Finding a mathematically exotic symmetry in a regular material we can find on Earth — well, it is mathematically beautiful and very interesting,” comments Robert Konik of Brookhaven National Laboratory in Upton, NY.

Scientists from England and Berlin began with chains of cobalt niobate, a magnetic material whose electrons have a preferred direction of spin — either up or down. The researchers chilled the cobalt niobate to 40 millikelvins (~273.1°Celsius) and applied a magnetic field. Without this external magnetic field, the spins of the electrons would all align in the same direction, as in an ordinary magnet. But an external magnetic field introduces a tension, and eventually the electrons prefer to align with that magnetic field instead of with their neighbors. The electron spins are associated with particle-like states, known as quasiparticles, in the system.

As the system approaches what’s known as a quantum critical point, blocks of quasiparticles begin changing orientation, says study coauthor Alan Tennant of the Helmholtz Association of German Research Centers in Berlin. That’s when the quasiparticles start resonating at mathematically intriguing frequencies. Two of the frequencies are in the ratio of the golden mean, the pleasing ratio of 1.618 often used in art and architecture. Ratios of the five frequencies found correspond to the complex E₈ Lie group symmetry.

“It is quite remarkable to see a material in the lab behaving with such perfection,” says Tennant. Perhaps this symmetry will also emerge in other physical systems and shed light on bigger questions, he says.
The Fundamental Group

If you have some kind of space (like the surface of a torus) and a basepoint, the **fundamental group** consists of paths that start and end at the same point. Two paths are considered the same if they can be continuously deformed into each other.

- Can you draw different group elements on a sphere or a torus?
- What do you think the identity element is going to be in this situation?
- Think about all the paths you could take walking around this room. Could that be made into a group?
Thank you!