How Do You Group It?

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Famous statements from science

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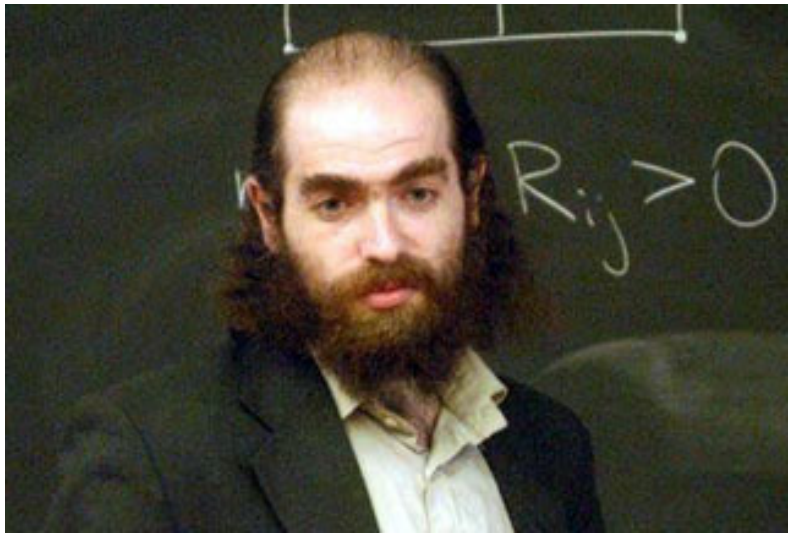

Grigori Perelamn

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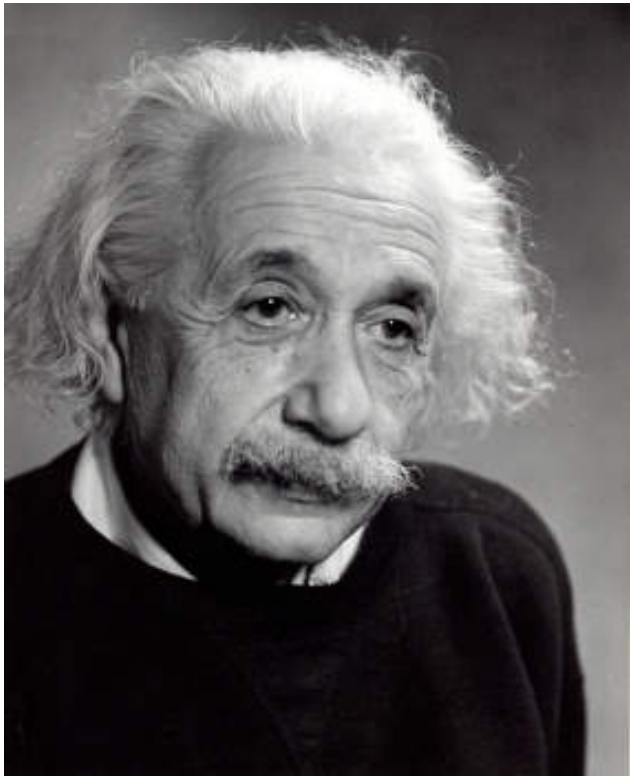

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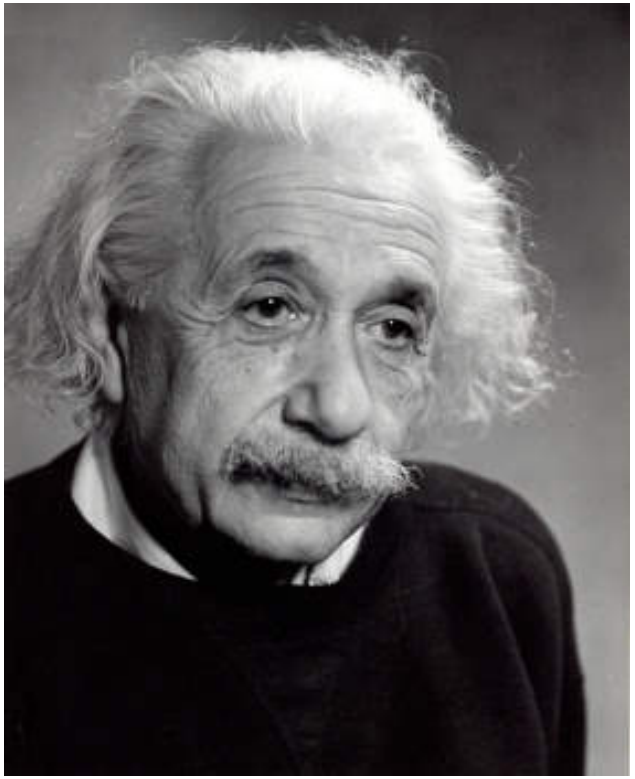

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What does "group" mean?

- (General) any collection of persons or things.
- (Chemistry) two or more atoms specifically arranged.
- (Music) a section of an orchestra.
- (Army) a flexible administrative and tactical unit

  ⋮
- **"Group" has a specific mathematical definition, too.**

So what IS a mathematical group?

Let's investigate a group that we have all seen before:

The Integers!

Our observations about integers

- Adding two integers together gives you another integer.
- It does not matter where the parenthesis go–you still get the same answer.
- Adding zero to any number does nothing.
- Every integer has an inverse, like $7 + (-7) = 0$

The definition of a group

A group G is a set of objects (called elements), and an operation $\circ$ (called composition) that obeys four rules. For all group elements a, b, and c we have:

Closure: $a \circ b$ is also in G.

Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

Identity: There's a special *identity* element e such that

$$e \circ a = a \circ e = a$$

Inverses: Every element a has an inverse a^{-1} such that

$$a \circ a^{-1} = a^{-1} \circ a = e$$

Which of these are groups?

- Even integers with addition?

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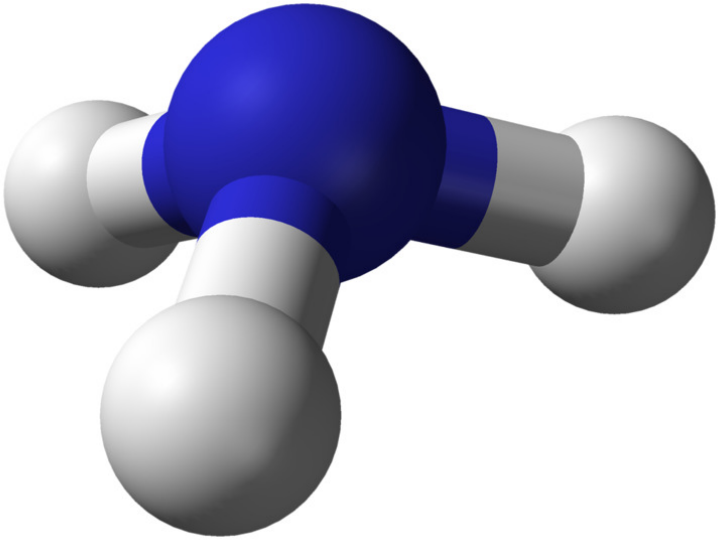
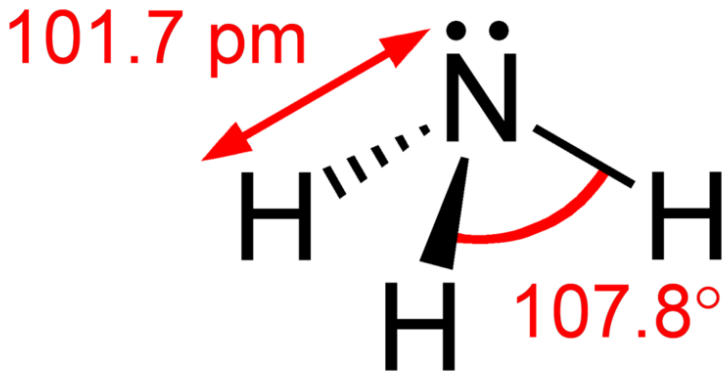
Ammonia





Rotations of an equilateral triangle

Counter-clockwise rotation	Letters in order	Name
Do nothing (0°)		R_0
Rotate once (120°)		R_1
Rotate twice (240°)		R_2
Rotate three times (360°)		R_3

1. If we rotate some more, what is going to happen with the letters?
2. Does ACB ever show up? Why or why not?

Counter-clockwise rotation	Letters in order	Name
Do nothing (0°)	ABC	R_0
Rotate once (120°)	CAB	R_1
Rotate twice (240°)	BCA	R_2
Rotate three times (360°)	ABC	R_3

The rotations themselves form a group. Does this make sense?

The rotations make a group

Let's check our understanding that the rotations are indeed a group.

1. How many distinct *rotational symmetries* does the triangle have?
2. How is this group different from the integers?
3. Inverses are pretty interesting in the rotation group.
   What is the inverse of R_0?
   What is the inverse of R_1?
   What is the inverse of R_2?

Rotations and reflections

Now let's consider a group made of both rotations and reflections. Let's look at how reflections act on the letters:

Reflection	Letters in order	Name
Reflect across line S_0		S_0
Reflect across line S_1		S_1
Reflect across line S_2		S_2

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Reflection	Letters in order	Name
Reflect across line S_0	ACB	S_0
Reflect across line S_1	BAC	S_1
Reflect across line S_2	CBA	S_2

Composition of group elements

	R_0	R_1	R_2	S_0	S_1	S_2
R_0	ABC		BCA		BAC	
R_1		BCA		BAC		
S_0	ACB				BCA	CAB

Composition of group elements

	R_0	R_1	R_2	S_0	S_1	S_2
R_0	ABC	CAB	BCA	ACB	BAC	CBA
R_1	CAB	BCA	ABC	BAC	CBA	ACB
S_0	ACB	CBA	BAC	ABC	BCA	CAB

the triangle VS the group

R_2 followed by S_0 is the same thing as S_1:

$$ABC \xrightarrow{R_2} BCA \xrightarrow{S_0} BAC$$

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We write:

$$S_0 \circ R_2 = S_1$$

The Cayley Table

	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0			S_0		
R_1			R_0		S_2	S_0
S_0	S_0		S_1			R_1

The Cayley Table filled out

	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0	R_1	R_2	S_0	S_1	S_2
R_1	R_1	R_2	R_0	S_1	S_2	S_0
S_0	S_0	S_2	S_1	R_0	R_2	R_1

An important observation:

- This group is not **Abelian**.

The Dihedral Group D_n

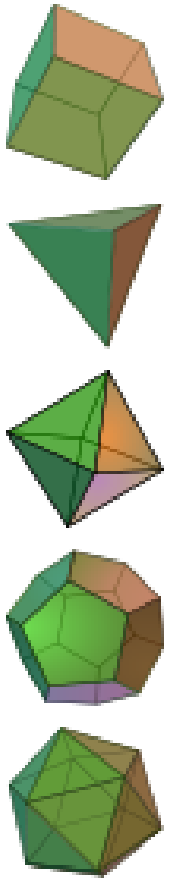
This group is called D_n, the Dihedral Group.

- The elements of D_n are rotations and reflections of a regular polygon with n sides.
- There are $2n$ elements total.
- D_n is finite.
- D_n is non-Abelian (for $n \geq 3$).
- D_n can be described by

$$\langle R, S \mid R^n = 1, S^2 = 1, SRS = R^{-1} \rangle$$

The Platonic Solids - regular convex polygons

- Cube
- Tetrahedron
- Octahedron
- Dodecahedron
- Icosahedrom



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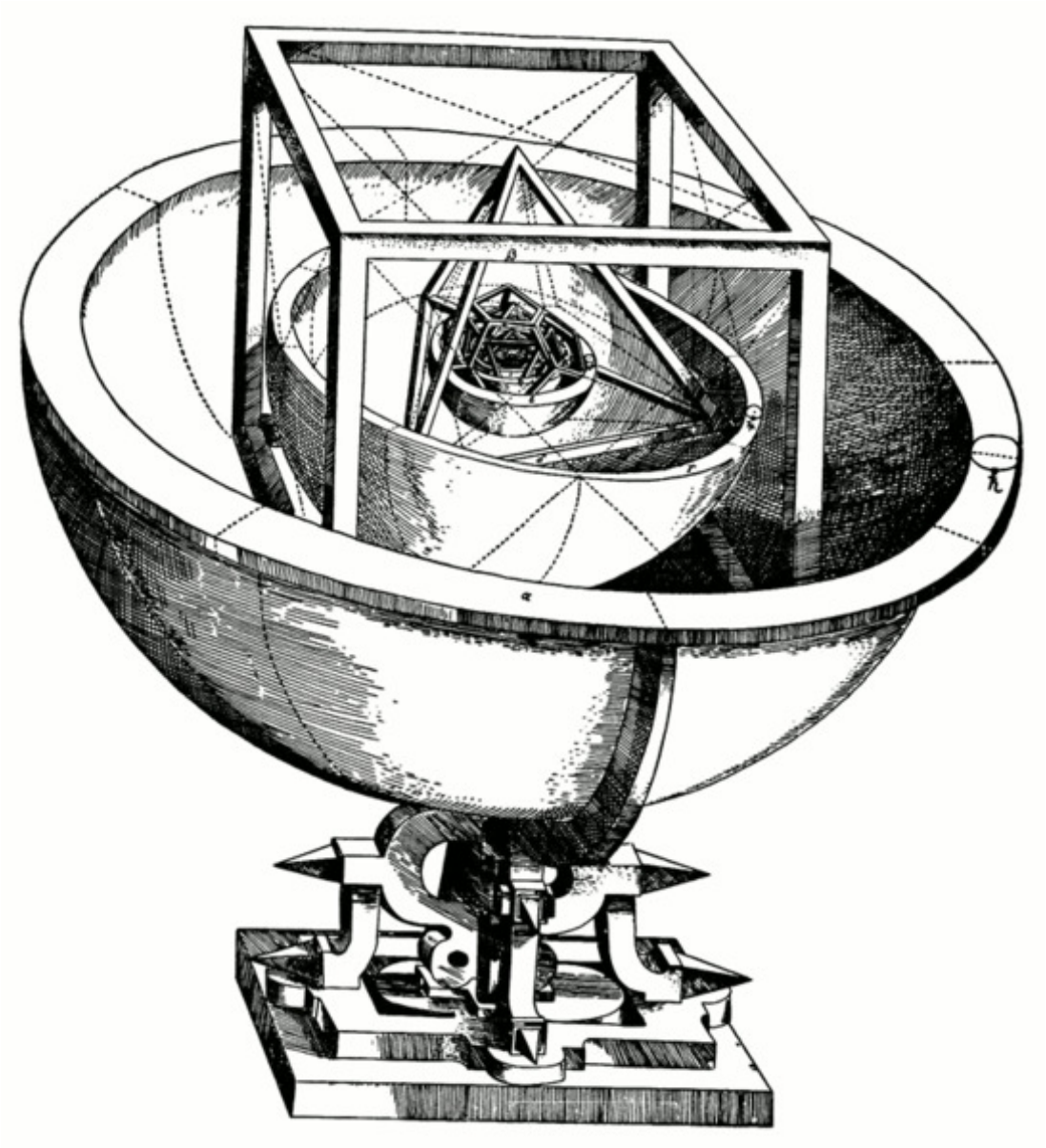
- 1595 A.D. - Johannes Kepler: Mercury, Venus, Earth, Mars, Jupiter, Saturn.

Kepler's Model of the Solar System



Science News

IN THE NEWS

Matter & Energy

"It is quite remarkable to see a material in the lab behaving with such perfection." —ALAN TENNANT

Elusive symmetry appears in nature

Complex E_8 patterns detected in supercold physical system

By Rachel Ehrenberg

A beautiful math emerges from the acrobatic flips of supercold atoms in a magnetic field.

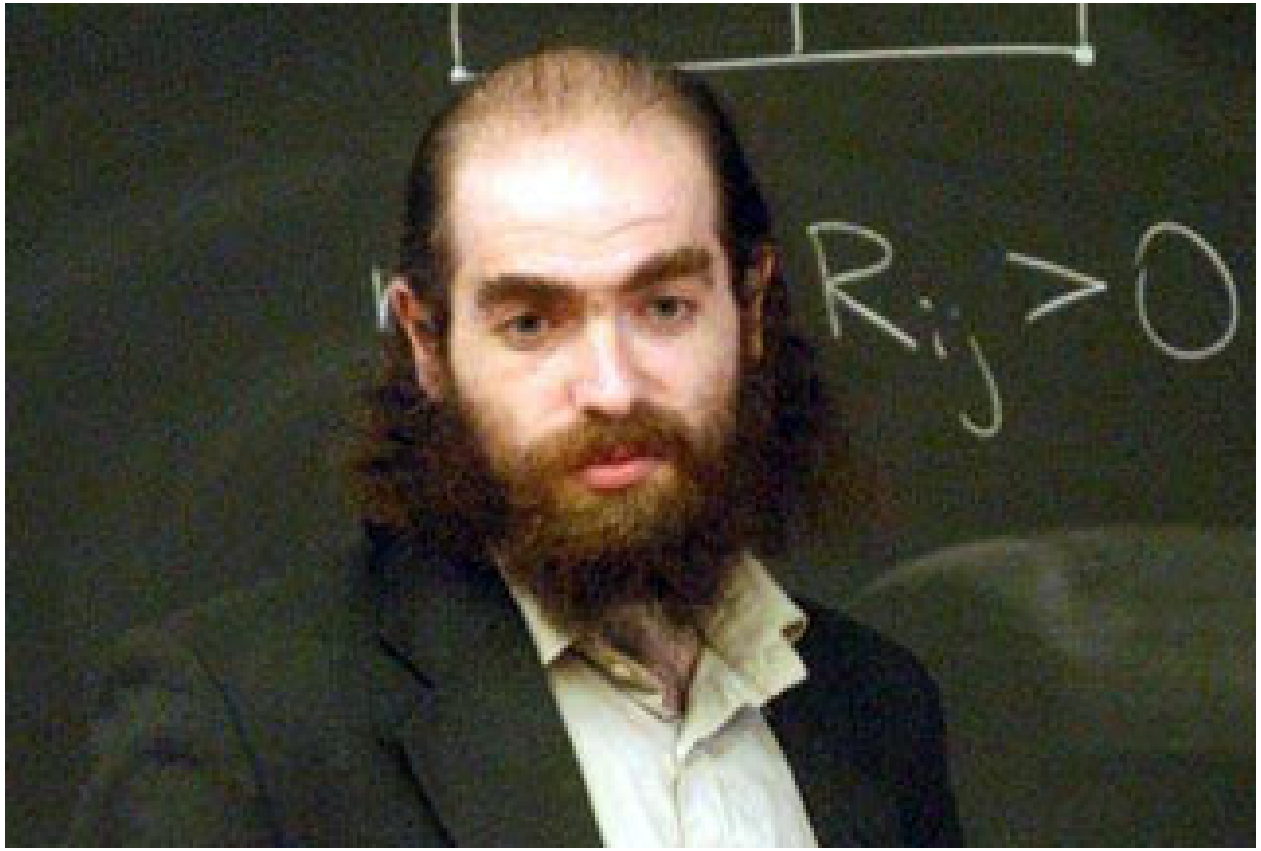
In the Jan. 8 *Science*, researchers report detecting an elusive, complex symmetry known as the E_8 Lie group, long analyzed on paper but never before seen in a physical system. The work suggests that this numerical grace may be hidden in other systems and may provide a mathematical link between quantum processes in matter and the physics of the cosmos.

"Finding a mathematically exotic symmetry in a regular material we can find on Earth — well, it is mathematically beautiful and very interesting," comments Robert Konik of Brookhaven National Laboratory in Upton, N.Y.

Scientists from England and Berlin began with chains of cobalt niobate, a magnetic material whose electrons have a preferred direction of spin — either up or down. The researchers chilled the cobalt niobate to 40 millikelvins (-273.1° Celsius) and applied a magnetic field. Without this external magnetic field, the spins of the electrons would all align in the same direction, as in an ordinary magnet. But an external magnetic field introduces a tension, and eventually the electrons prefer to align with that magnetic field instead of with their neighbors. The electron spins are associated with particle-like states, known as quasiparticles, in the system.

As the system approaches what's known as a quantum critical point, blocks of quasiparticles begin changing orientation, says study coauthor Alan Tennant of the Helmholtz Association of German Research Centers in Berlin. That's when the quasiparticles start resonating at mathematically intriguing frequencies. Two of the frequencies are in the ratio of the golden mean, the pleasing ratio of 1.618 often used in art and architecture. Ratios of the five frequencies found correspond to the complex E_8 Lie group symmetry.

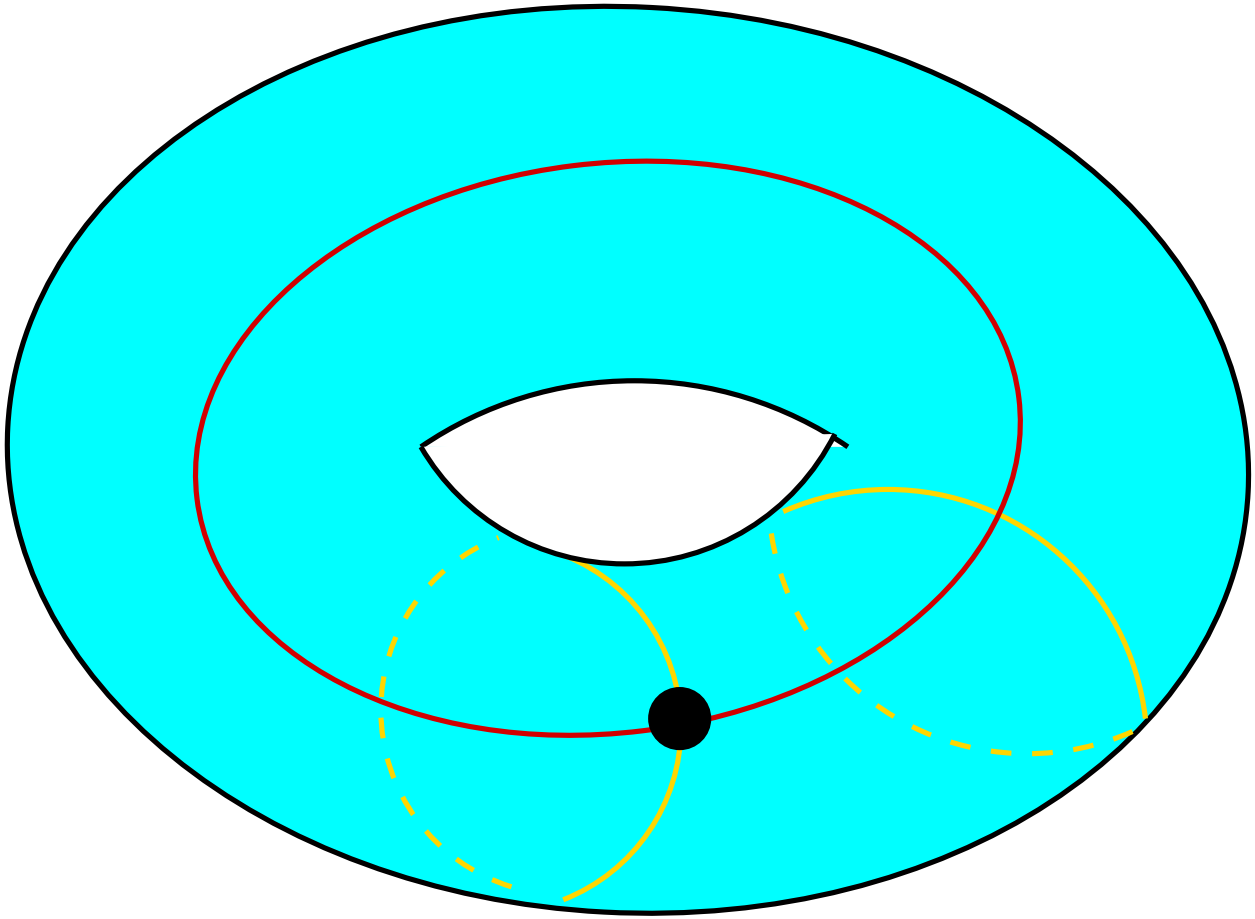
"It is quite remarkable to see a material in the lab behaving with such perfection," says Tennant. Perhaps this symmetry will also emerge in other physical systems and shed light on bigger questions, he says.


$R_{ij} > 0$

The Fundamental Group

If you have some kind of space (like the surface of a torus) and a basepoint, the **fundamental group** consists of paths that start and end at the same point. Two paths are considered the same if they can be continuously deformed into each other.

- Can you draw different group elements on a sphere or a torus?
- What do you think the identity element is going to be in this situation?
- Think about all the paths you could take walking around this room. Could that be made into a group?



Thank you!