Math Rocks

Dave Auckly

The aim of this project is to explain the mathematics that leads to the classification of the 32 basic crystal systems. A crystallographic group is a discrete group of isometries of $\mathbb{R}^3$ that contains three independent translations. There are 230 different crystallographic groups. Minerals each have a crystallographic type corresponding to the symmetry group of the underlying ideal crystal structure.

Any isometry of R^3 takes the form $x \mapsto Ax+b$ for some orthogonal A. The crystal classes correspond to the point groups of the crystallographic groups. If Γ is a crystallographic group, the associated point group is

$$P_\Gamma = \{A | Ax + b \in \Gamma\}\,.$$

The symmetry elements that can appear in a point group are, n-fold rotations (denoted by an n with no star to the left), n-fold dihedral (denoted by n with a star to the left), reflection denoted by $*$, and order $2n$ roto-inversion denoted by nX. The notation scheme we are describing is due to Thurston. By convention we record all rotational and dihedral symmetry elements in increasing order such as 233 and $*234$. These symmetries are the steps in our *holiday dance*.

To list all possible point groups (classify them), one notices that the point groups act on the sphere and considers the quotient "pillow case" or "holiday gift". Clearly these are all *spherical quotients*. These quotients must satisfy the so called *crystallographic restrictions*. Two symmetry types imply the existence of even more symmetry – $2*4$ and $2*6$. By using the Euler characteristic, one can derive a complete list of the 32 point groups. The following problems outline this process.

1. Show that the tear drop packages denoted by n for $n > 1$ cannot be spherical quotients. Do the same for the asymmetric footballs nm for $n \neq m$ and the versions with one added reflection $*n$ and $*nm$.

2. The **crystallographic restrictions** are that the only allowed rotational or dihedral symmetry elements are n and $*n$ for $n = 2, 3, 4, 6$ and the only allowed roto-inversions have order 2, 4 and 6.

3. Show that any symmetry group of the sphere containing one of type $2*4$ ($2*6$) must actually contain one of $*24$ ($*26$).

   (a) A "wrapping paper" group is the $2D$ analogue of a crystallographic group. Show that such a group cannot contain an order 5 rotation by considering how such a rotation could be combined with a "shortest" translation.

   (b) Similar arguments can be used to show that only rotations n for $n = 2, 3, 4, 6$ are possible in "wrapping paper" groups. Assume this is true and use it to derive the crystallographic restrictions by showing that any symmetry element in a crystallographic group violating these restrictions would lead to a unallowed rotation in a "wrapping paper" group.

4. The Euler characteristic is a wonderful tool for limiting the potential crystal classes. One important property is that it is multiplicative under covers.

   (a) Glue the two boundary radii on half of a disk to create a cone. Given that this is the quotient of a disk by an order 2 rotation, what does the Euler characteristic of this cone have to be? What is the angle around this cone point? Does this match with the arguments that we made to derive the invariance of the Euler characteristic? Can you modify those arguments to take this phenomena into account?

   (b) We can construct spherical quotients by adding symmetry elements to a sphere. The costs of the various symmetry elements are as follows: n with no star to the left costs $(n-1)/n$; n with a star to the left costs $(n-1)/2n$; a star costs 1; X costs 1; $2X$ costs $3/2$; $3X$ costs $5/3$. Explain where these costs come from.

   (c) Explain the difference between $2*$ and $*2$. What about $2*3$ and $*23$?

   (d) Since the Euler characteristic of a sphere is 2 and Euler characteristics are multiplicative, the Euler characteristic of any spherical quotient must be positive. Write a list of all possible symmetry combinations with total cost less than the 2 unit starting point, that arise as spherical quotients and satisfy the crystallographic restrictions and cross out $2*4$ and $2*6$ (there will be 32).

   (e) To finish this classification we need to know that each of these 32 symmetry combinations is possible (we may have missed some restrictions). Look at the various ornaments (Rock crystals) and identify the associated symmetry combination. We have to check that these assemble into complete crystals.

   (f) Make your own rock crystal ornament from the developed patterns.

The same Euler characteristic techniques may be used to classify all "wrapping paper" groups. This is addressed in the other hand-out. The key idea is that since each such group contains two independent translations, each group can be used to wrap a torus (king cake for the holiday?), thus any quotient "package" must have Euler characteristic zero since this is the Euler characteristic of a torus.