Fabric patterns and the shape of space

Most everyone has seen patterns on fabric. The symmetry of such patterns has been classified. More information can be found in the book by Conway, Burgiel and Goodman-Strauss. That book is the source of the images on this worksheet. There are several basic types of symmetry. These are listed below with a signature and a cost.

Mirror line A mirror line or reflection symmetry is a line along which the pattern can be folded and have both halves of the pattern match. The signature associated to such a line is $*$ and the cost is 1.

Rotational A point of rotation is one of the simplest types of symmetry. At such a point the entire pattern can be rotated $1/n$ of the way around and it will exactly match itself. There are no lines of reflection through such a point. The signature associated to such a symmetry is n (written before any star) and the cost is $(n-1)/n$.

Dihedral A point with dihedral symmetry of order $2n$ combines a rotational symmetry of order n with a mirror line. The signature associated to such a symmetry is an n after the star (which much exist) and the cost is $(n-1)/n$. See figure 7.

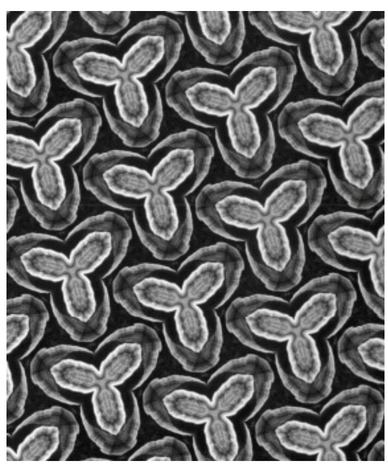
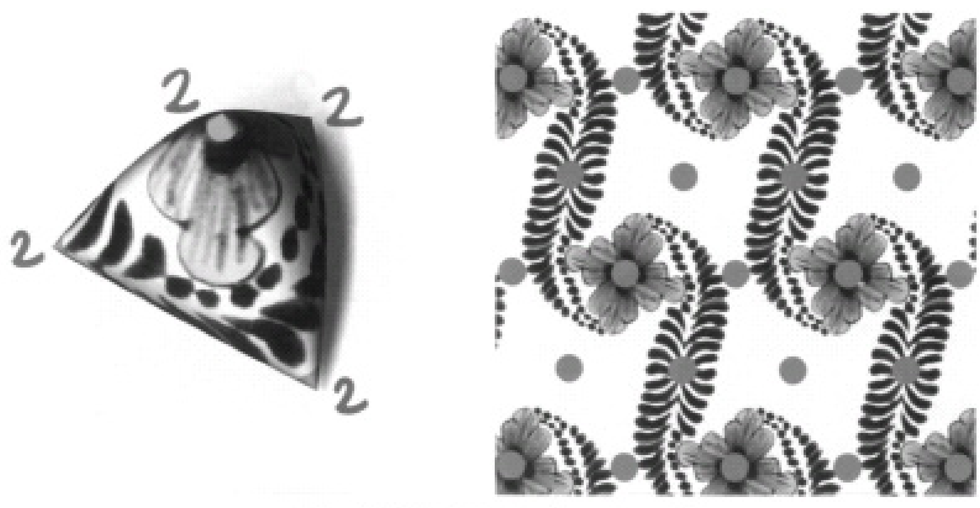


Figure 7: 3 star 3 fabric

Glide reflection A glide reflection occurs when the fabric can be translated some distance along a line and then be reflected and have the pattern match (the translation must be there – if it matches without, it is a mirror line). The signature associated to such a symmetry is $\times$ and the cost is 1.

Translation If the fabric has none of the above symmetries, but has two independent translational symmetries, we say it only has translational symmetry. This has signature $\circ$ and costs 2 units.

Given a piece of fabric with a repeating pattern, it is possible in principle to fold the fabric into a smallest 'pillow' so that the pattern matches perfectly as one goes around the 'pillow' but does not repeat on the 'pillow'. Such a 'pillow' is called the orbifold associated to the wallpaper (or fabric) pattern. In this way we can describe interesting shapes by describing the associated pattern. See figure 8.



The orbifold for **2222** is simply a sphere with four cone points.

Figure 8: Fabric 'pillow' or orbifold

In fact we only say a piece of fabric has a wallpaper pattern when the associated orbifold is finite. There are exactly 17 wallpaper patterns, and they can be determined from the fact that the total cost of all symmetries is 2 units. There is a similar theory for three dimensional patterns. The resulting patterns are called crystal patterns (they originally were studied in the context of crystallography in geology.) Some scientists are looking for the pattern in the 'fabric' of space to try to determine the shape of the Universe. See http://www.ams.org/notices/200406/fea-weeks.pdf (a bit technical) or the book by Weeks for more information.

1. In a game on 'the price is right' a contestant is given a certain amount of money (say \$2.00) and must go shopping trying to spend as close to \$2.00 as possible. There are cans of food C_n that cost $(n-1)/n$ each. There is a tree that costs 2. There is a box of special K that costs 1, there is a bottle B that cost 1, and there are refills for the bottle that can only be bought if you buy the bottle. The refills R_n cost $(n-1)/(2n)$. In how many ways can one spend exactly \$2.00? (List the possibilities in an organized way.)

2. For each of the wallpaper pattern below, mark representative rotational and dihedral fixed points, write the signature of the pattern, and draw a sketch of the quotient pillowcase.

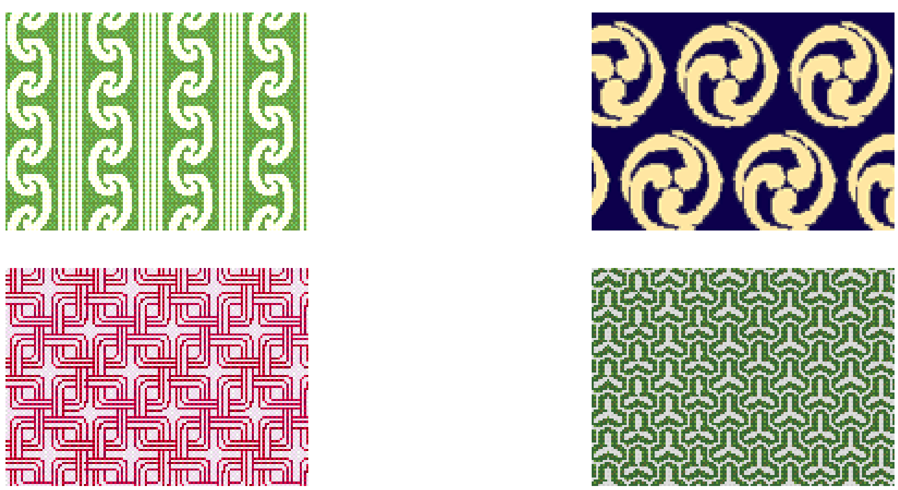
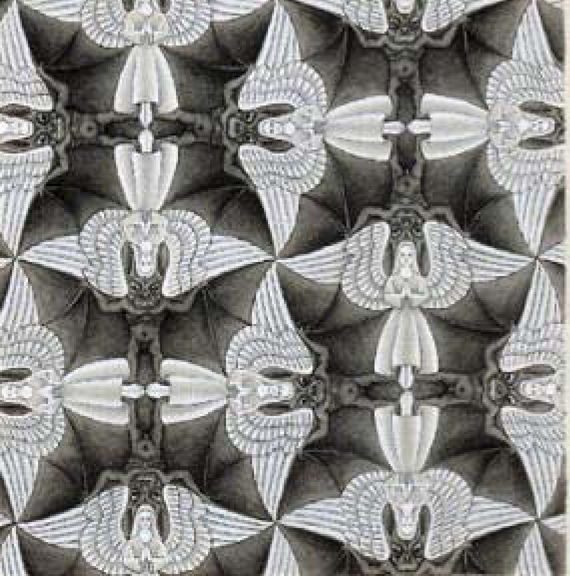


Figure 9: Four fabrics



LCD TV ...

1. Not every tiling of the plane is periodic. Interesting aperiodic tilings may be constructed via recursion. Use the recursive rule replacing every triangle with a collection of triangles as pictured below, to compute the fourth term in the sequence. A part of a tiling constructed in this way is displayed to the right. If the short side of the original triangle has length 1, How long is the long leg?