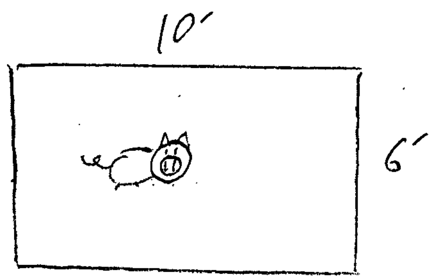
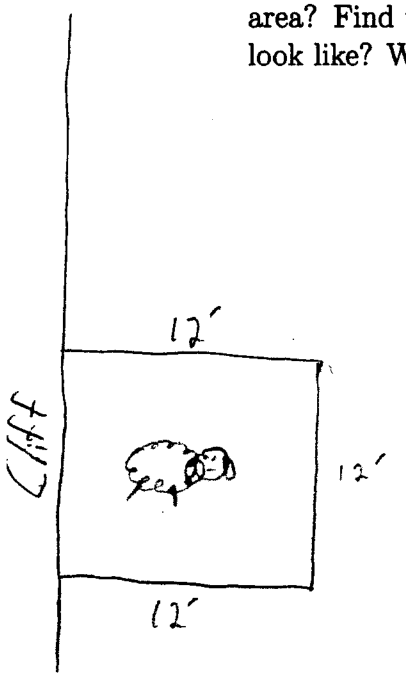
Optimization

-1. Farmer Bob wants to build a rectangular pig pen using 32 feet of fence, and he wants to *maximize* the area of the pen. He built the pen shown. Did he maximize the area? Find the shape he should have made by trial and error. What does this shape look like? What is the maximum area of the pen?



0. Now Farmer Bob wants to build a rectangular pen for his sheep. He has 36 feet of fence to use this time, but he's going to build it against the side of a cliff, so that he doesn't need to use fence on that side. He built the pen shown. Did he maximize the area? Find the shape he should have made by trial and error. What does this shape look like? What is the maximum area of the pen?



AM and GM

To find the *arithmetic mean* (AM, a.k.a. the average) of a list of numbers, we add them all up and divide by the total number of terms. To find the *geometric mean* (GM) of a list of numbers, we *multiply* them all together and then take the *root* corresponding to the total number of terms. In other words, if we call our numbers $x_1, x_2, \ldots, x_n$, so the total number of terms is n, then we get
AM= $\frac{x_1+x_2+\cdots+x_n}{n}$,
and
GM= $\sqrt[n]{x_1 x_2 \ldots x_n}$

Find the AM and the GM of the following triplets.

(a) 1, 5, 25

(b) 1, 2, 4

(c) 3, 3, 3

Do you notice a relationship between the AM and the GM?

Optimization

The following optimization problems are loosely ordered from easy to hard. For each problem, it's a good idea to do the following.

a. Draw a picture if possible.

b. Label your variables.

c. Write down what you want to *maximize* or *minimize*.

d. Write down your *constraint*.

e. Set up the AM-GM inequality. It's usually easiest to make the AM side look like what you want.

f. Manipulate your inequality to get the information you need.

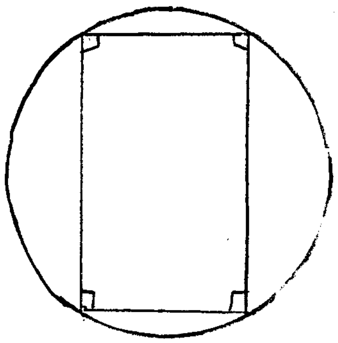
1. Two positive numbers have to add up to 100. What is the biggest you can make their product?

2. Farmer Bob wants to build a rectangular sandbox for his kids to play in (his human kids, not his baby goats – they would eat the toys). He has enough sand to cover an area of 100 square feet, so that's what he wants the area of the sandbox to be. He will have to install a special kind of fence around the sandbox, which prevents the cats from getting into the sandbox. This fencing is very expensive, so he wants to *minimize* the perimeter of the sandbox. How should the sandbox be shaped?

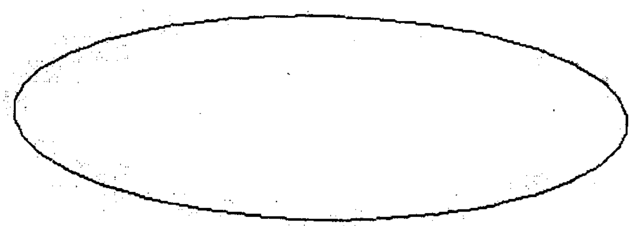
3. Farmer Bob needs to build a box out of ply-wood and paint it red. It's going to be shaped like a rectangular prism. He only has enough paint to cover 66 square inches, so he wants the surface area to be 66 square inches. How can he maximize the volume?

4. A company wants to sell rectangular boxes filled with magic beans. Each box needs to hold a volume of 5 cubic feet of magic beans. They want the box to have the smallest possible surface area, so that the boxes are cheap to make. How should they do it?

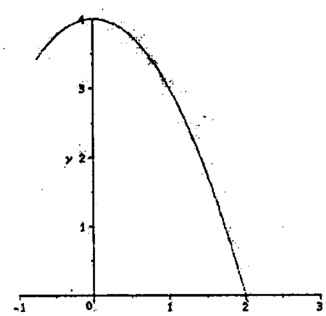
5. The picture below shows a rectangle *inscribed* in a circle with radius 5. What is the rectangle with the largest area that can be inscribed in this circle?



6. What is the rectangle with largest area that can be inscribed in the ellipse shown, whose equation is $\frac{x^2}{9} + y^2 = 1$?



7. (tricky) Find the largest area of a rectangle whose bottom side is on the x-axis, whose left side is on the y-axis, and whose top-right corner is on the curve $y = 4 - x^2$.



8. (tricky) Find the largest volume of a cylinder that can be inscribed in a sphere of radius 10.

9. (trickier) Find the smallest possible value of $\frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}}$.