How to Solve It

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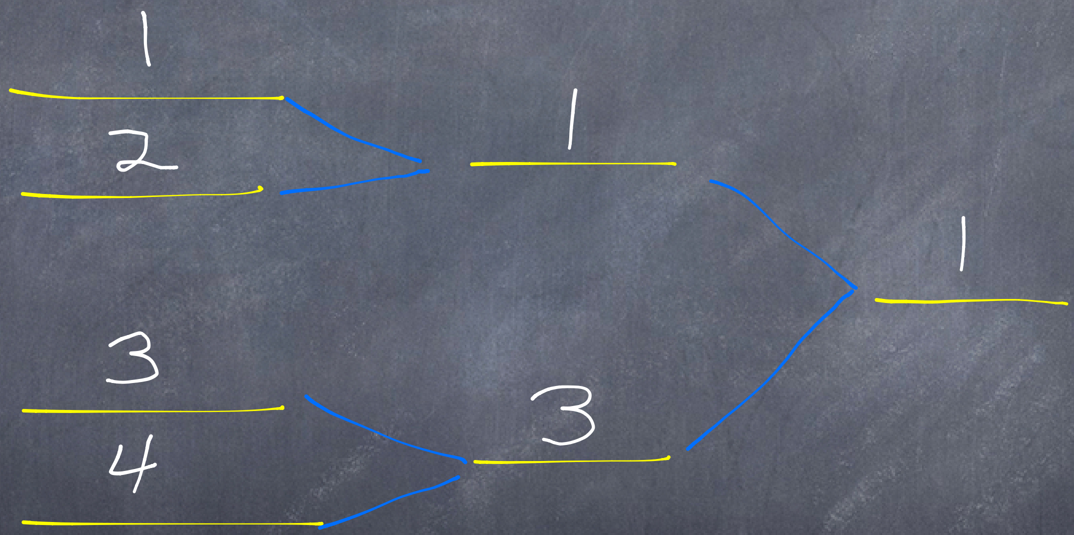
We’ll do lots of fun problems today, and we’ll learn a bit about problem-solving.

To keep this fun and productive for everyone, a few suggestions:

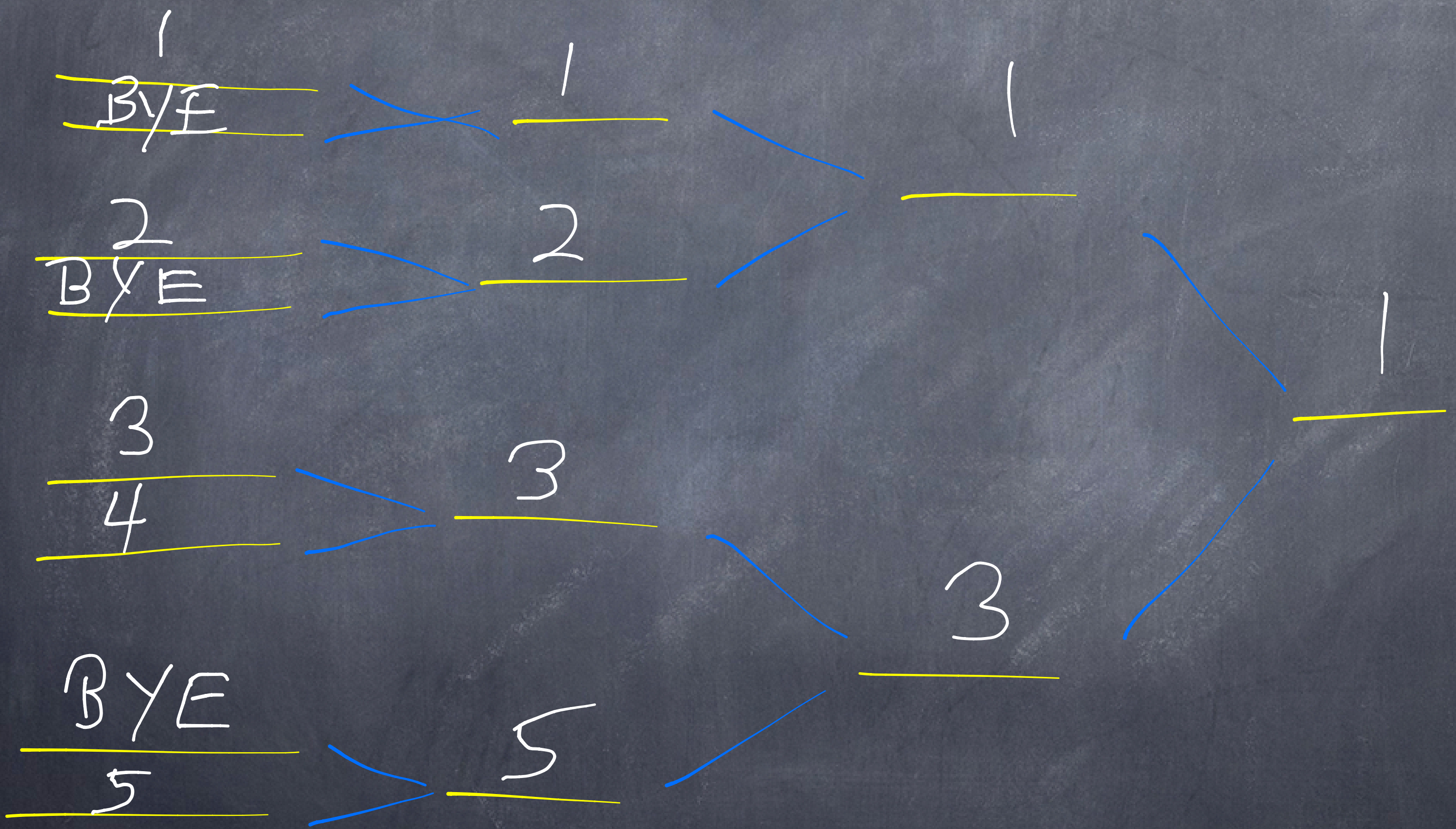
- When you have a solution, wait before sharing so others can have time to work.
- If you’ve heard a problem before, let others experience the joy of finding the solution!

Scheduling a tennis tournament

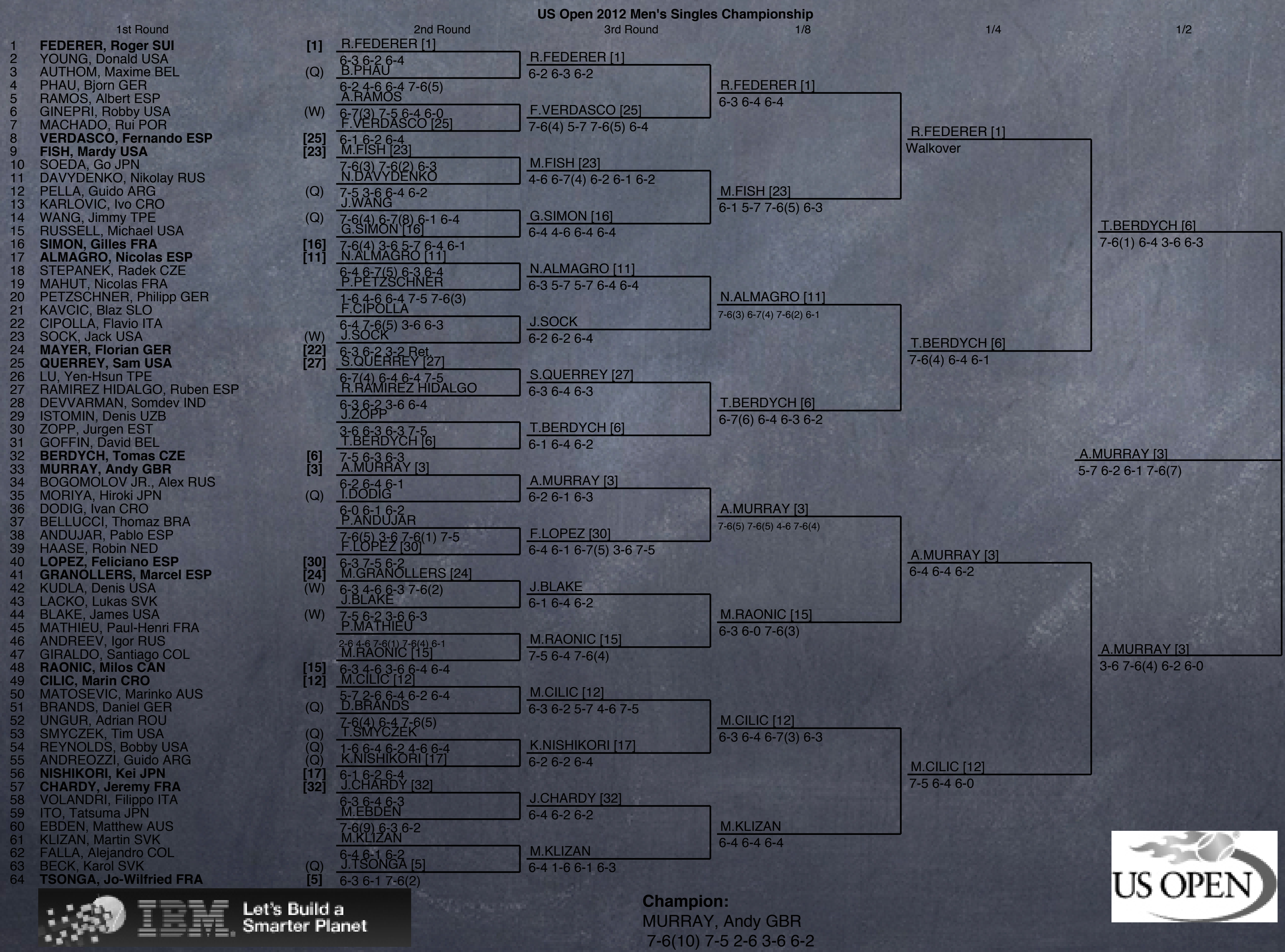
A single elimination tournament has a bracket, such as that used in the NCAA March Madness. With 4 players:

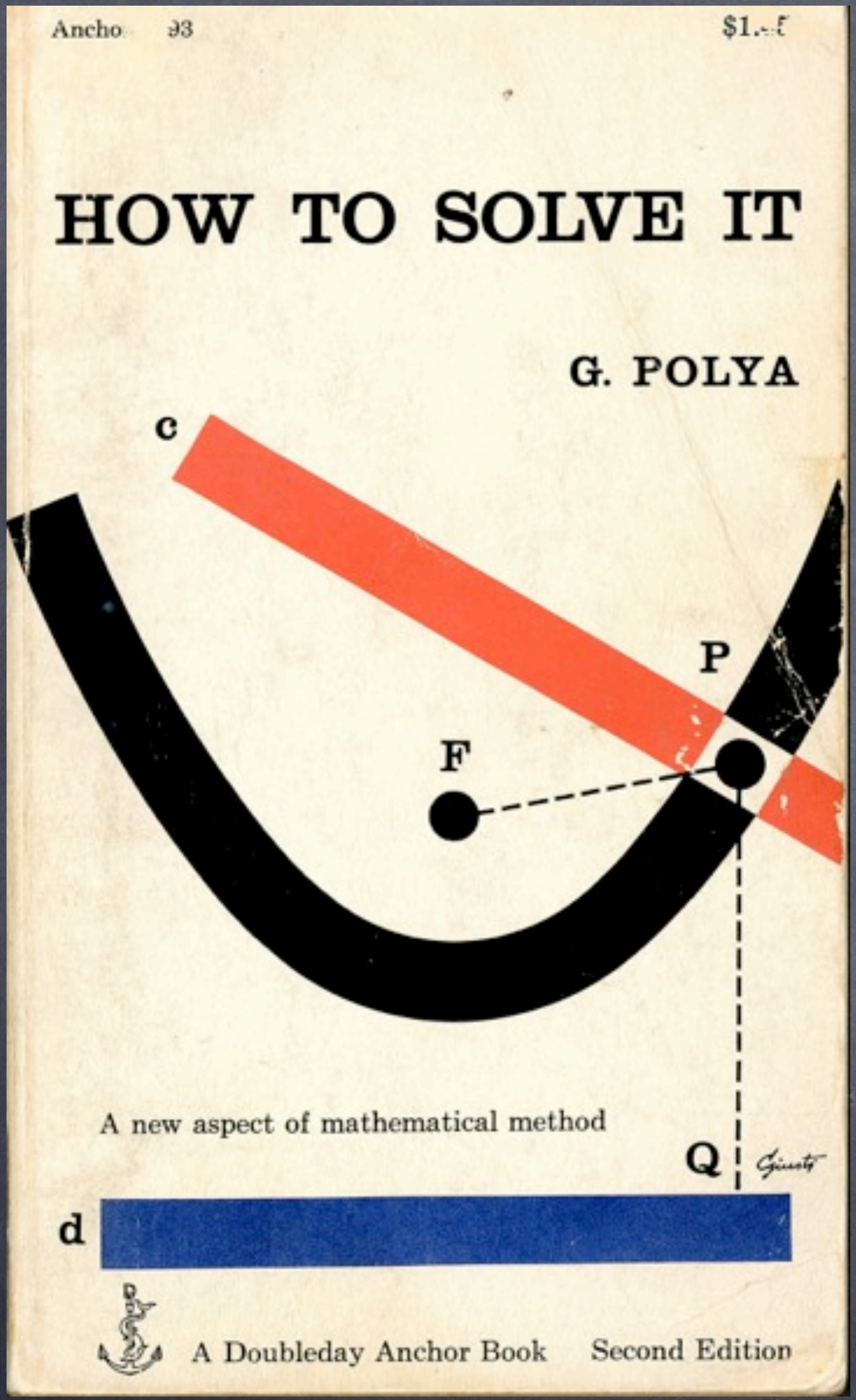


If there is an "uneven" number of players, then Byes are used so that there is an "even" number in the 2nd round:



Problem: You are the director of a tennis tournament with 50 players. You are in charge of scheduling the courts. How many total matches will there be in the tournament?





George Pólya
(1887–1985)

Today we scratch the surface of *heuristic*, the *methods and rules of discovery and invention*. This applies to problem solving...and to research.

How to attack a problem? Inspiration may come quickly. If not, technique helps!

Today we will learn a few techniques and have some fun using them in elementary problems. One of the main techniques is experience, and that can only be acquired through constant problem-solving.

I highly recommend Pólya’s book as a good starting point.

Take breaks. Take long walks. Let your mind wander. Here’s a famous story of Henri Poincaré (1854–1912):

Just at this time I left Caen, where I was then living, to go on a geologic excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the result at my leisure.

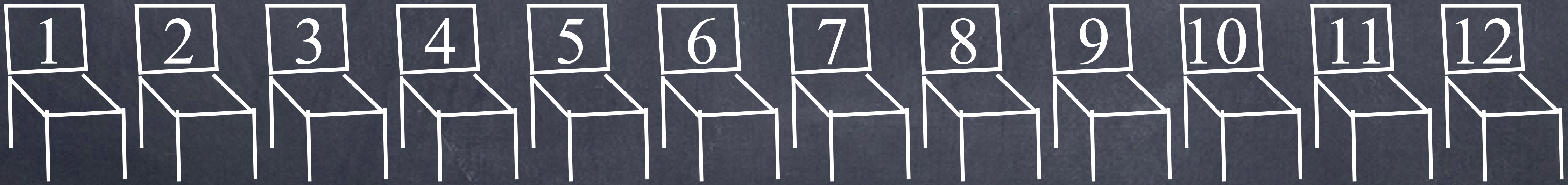
The basic technique today is: Vary the problem

There are many variations: generalization, specialization, decomposition, auxiliary elements, ...

Generalization and specialization are easy to do if there is a number (or numerical variable) in the problem.

Here is a problem from a contest at the new math museum in New York:

Problem: 12 people are seated in a row. They stand, permute, and sit again. In how many ways can they rearrange themselves if each person moves at most one seat?



When you solve a problem, it is important to look back. Here's a list of questions from Pólya:

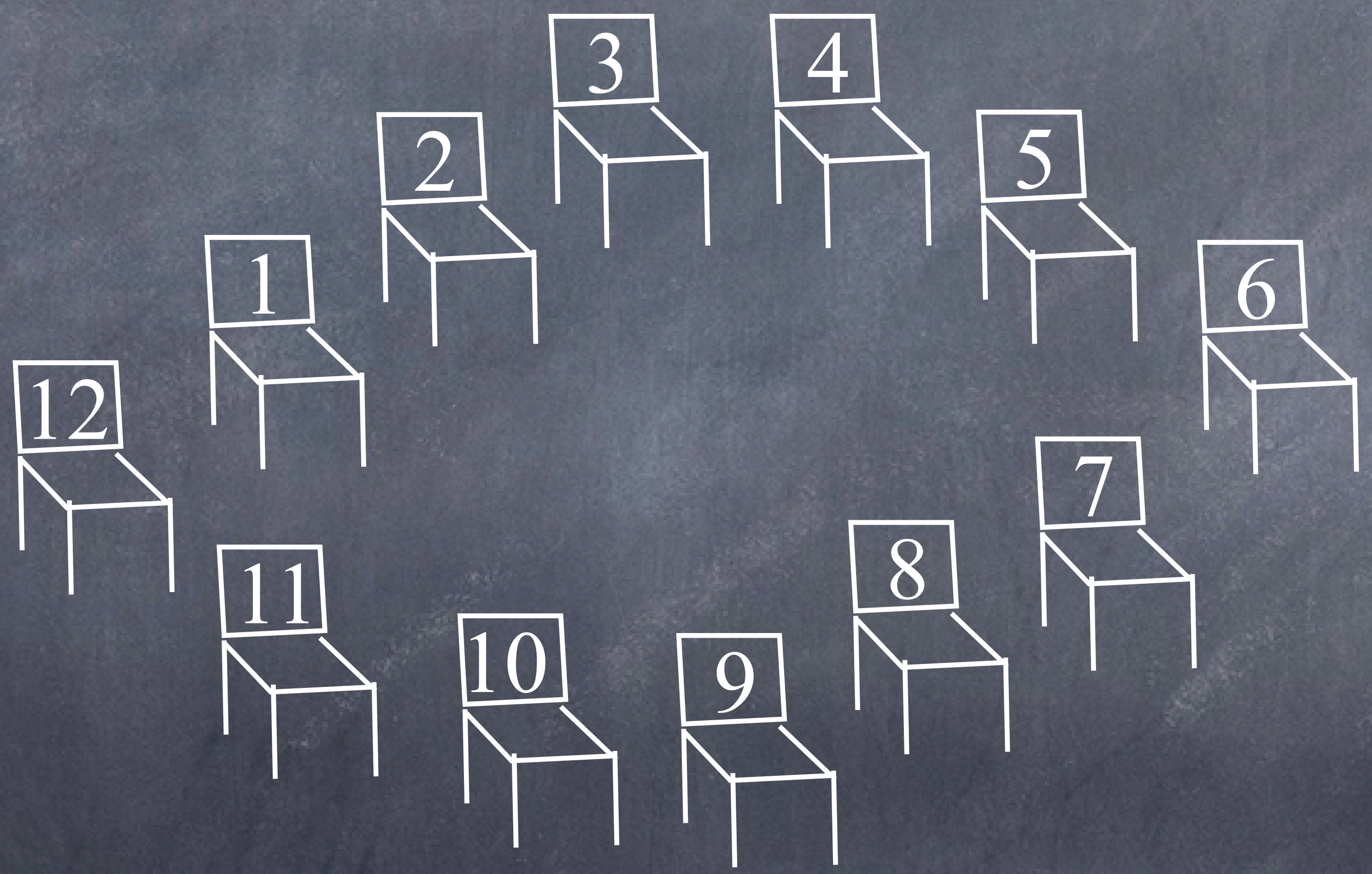
- Can you check the result? Can you check the argument?
- Can you derive the result differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

And, on our theme of variation:

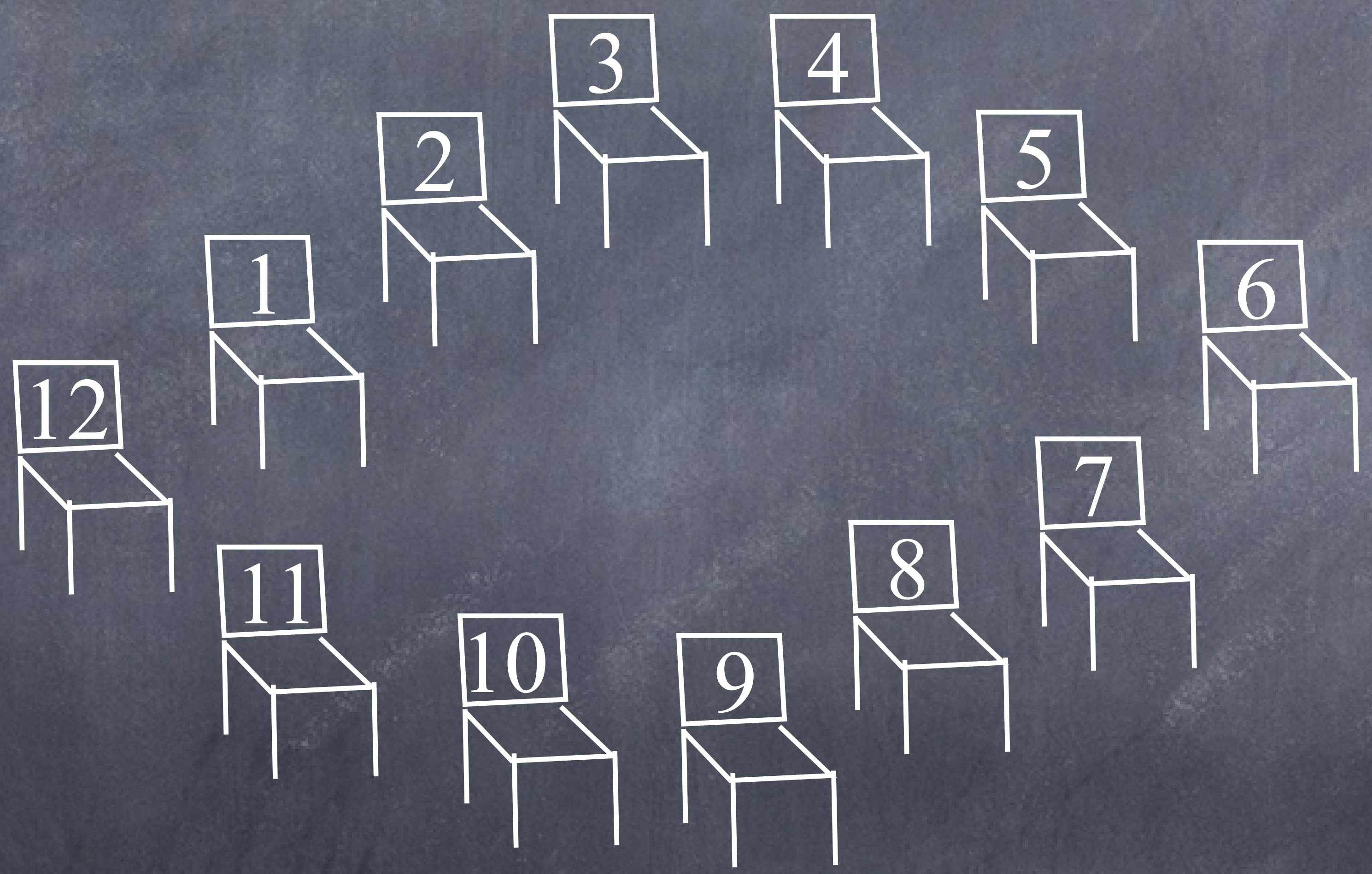
- Can you pose a new problem suggested by the previous problem, result, or method?

This is how you begin the "research patter".

Problem: 12 people are seated in a *circle*. They stand, permute, and sit again. In how many ways can they rearrange themselves if each person moves at most one seat?



Problem: Here's another variation. 12 people are seated. They stand, permute, and sit again. In how many ways can they rearrange themselves if each person does *not* sit where he/she started?



Problem: A tropical island near Bora Bora has 60 inhabitants. Each has blue eyes, but somehow doesn't realize it. That's a good thing since local custom demands that on the day a person finds out he/she has blue eyes, then he/she must evacuate the island at midnight. Another local custom is the daily noon lunch gathering, where all of the inhabitants mingle and see each other. Now one beautiful May 1 a stranger arrives on the island and, at the noon gathering, announces ``Someone on this island has blue eyes''. For the first time each inhabitant begins to think seriously about the possibility that his/her own eyes are blue. What happens?

Problem: How can we estimate the volume of water in a reasonably small, irregularly shaped, shallow pond?

This problem is open-ended and there is not a definite right answer. (Previous problems have a right answer, but not a “right” method of solution.)

Still, problem-solving techniques apply.

In this case, we introduced an auxiliary element into the problem. Often that is a good step, but it takes some inspiration to find the right one.

Let’s look over Polya’s summary chart:

HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First.

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

CARRYING OUT THE PLAN

Third.

Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth.

Examine the solution obtained.

Can you *check the result?* Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

These same principles apply on a grander scale.

Consider how the notion of ‘space’ has evolved and is still evolving:

The Greeks codified ideas about 2- and 3-dimensional spaces. (They developed the basic mathematical framework of definitions, axioms, and theorems.)

Almost immediately a question: What happens if we drop the parallel postulate?

Eventually (early 1800s) new geometries emerged: spherical and hyperbolic geometry.

The sphere is defined by a polynomial equation:

$$x^2 + y^2 + z^2 = 1$$

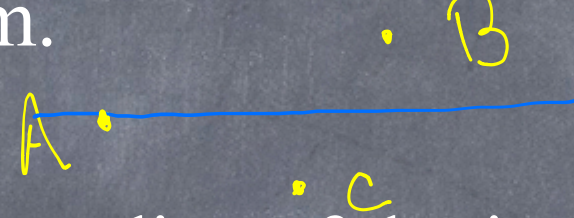
What about geometric shapes defined by different polynomials? Different functions? Different numbers of dimensions?

More radical ideas: What about geometry over different number systems? What if x and y are complex numbers? Numbers in clock arithmetic?

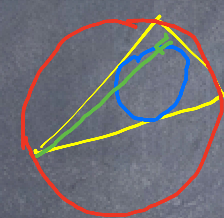
Modern geometry has even more radical notions of space, *all the result of the "research patter"*.

What is surprising and intellectually satisfying is how these ideas—even many outrageous ones—apply to science and technology.

Problem: Given points A, B, and C in the plane, construct the line containing A which is equidistant from B and C and passes between them.



Problem: In a triangle let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the longest altitude. Show that $H \geq r + R$. (There is a nasty twist here...)



Problem: Two trains 200 miles apart are moving toward each other; each one is going at a speed of 50 miles per hour. A fly starting on the front of one of them flies back and forth between them at a rate of 75 miles per hour. It does this until the trains collide and crush the fly to death. What is the total distance the fly has flown?

John von Neumann (1903–1957)

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The fly actually hits each train an infinite number of times before it gets crushed, and one could solve the problem the hard way with pencil and paper by summing an infinite series of distances. The easy way is as follows: Since the trains are 200 miles apart and each train is going 50 miles an hour, it takes 2 hours for the trains to collide. Therefore the fly was flying for two hours. Since the fly was flying at a rate of 75 miles per hour, the fly must have flown 150 miles. That's all there is to it.

When this problem was posed to John von Neumann, he immediately replied, "150 miles."

"It is very strange," said the poser, assuming the professor had immediately seen the simple solution, "but nearly everyone tries to sum the infinite series."

"What do you mean, strange?" asked Von Neumann. "That's how I did it!"

Summary

What have we learned?

- Problem-solving techniques are helpful
- Develop the "mathematical patter" as a mode of thought
- When you finish a problem you're not finished
- Let your mind work subconsciously
- Have fun with mathematics!