Mathematics of voting

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Saturday Morning Math Group

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[1] Thanks to TED Ed, Wikipedia contributors, Los Angeles Math Circle `http://www.math.ucla.edu/~radko/circles/` and Lian Smythe `http://www.math.cornell.edu/~ismythe/S15_1340.html`, whose material I used in these slides

Griffindor prefect elections

1. Harry Potter (P)
2. Hermione Granger (G)
3. Ronald Weasley (W)
4. Neville Longbottom (L)
5. Seamus Finnigan (F)

$31 : P > L > W > F > G$

$20 : G > L > W > F > P$

$19 : L > W > F > G > P$

$16 : F > W > G > P > L$

$14 : W > F > L > G > P$

Who should win?

Exercise. Construct 5 different fair voting methods, each of which produces a different winner.

Some fair voting methods

1. **Plurality.** Everybody votes for their prefereed candidate. The candidate with the most votes wins. (P)
2. **Top-two run-off.** First, each voter votes for their preferred candidate. We select the two candidates with the most votes, and we have a second round only with them. The candidate among the two with the most votes wins. (G)
3. **Borda Count.** If there are N candidates, each voter assigns N − 1 points to their preferred candidate, N − 2 votes to their second preferred candidate, and so on, to 0 points for their least-preferred candidate. The candidate with the most points wins. (W)
4. **Instant Run-off Voting.** Each voter votes for their favourite candidate. The candidate with the least votes is eliminated. Repeat until there is only one candidate left. (F)
5. **Survivor.** Each voter votes for their least favourite candidate. The candidate with the most votes is eliminated. Repeat until there is only one candidate left. (L)

What makes a voting method fair?

1. **Majority criterion.** If a majority (i.e. more than 50%) of the voters have X as top choice, then X wins.
2. **Majority–loser criterion.** If a majority of voters have X as their last choice, then X does not win.
3. **Pareto criterion.** If every voter prefers X to Y, then Y does not win.
4. **Condorcet criterion.** Assume there is a candidate X such that, for any other candidate Y , X would win against Y in a head-to-head race just between the two of them. Then X wins.
5. **Independence criterion.** If an election is held and a winner is declared, this winning candidate should remain the winner in any recalculation of votes as a result of one or more of the losing candidates dropping out

Exercise. For each of the five voting systems above, which ones of the five criteria do they pass or do they fail? For example, Plurality passes majority criterion. On the other hand, Survivor fails the majority criterion because we can construct a scenario in which a candidate with more than 50% of the votes loses the election.

Arrow's criteria

A **preference profile** is the list containing the rankings of candidates by each of the voters (like in the case of the Griffindor elections).

A **voting method (social choice function)** is a method (procedure) that decides on the winner, based on the preference profile (like Plurality, Borda Count or the Survivor).

Arrow's criteria:

1. **unanimity (Pareto)** criterion: if everybody prefers X to Y, then Y cannot win.
2. **independence** criterion: If an election is held and a winner is declared, this winning candidate should remain the winner in any recalculation of votes as a result of one or more of the losing candidates dropping out

After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Arrow's impossibility theorem

If there are 3 or more candidates, then any voting method which satisfies the Pareto and independence criteria is necessarily **dictatorial.**

(Dictatorial means that there exists one voter whose top choice is immediately declared to be the winner, no matter what the others' preferences are.)

Kenneth Arrow
Nobel Prize in Economics, 1972

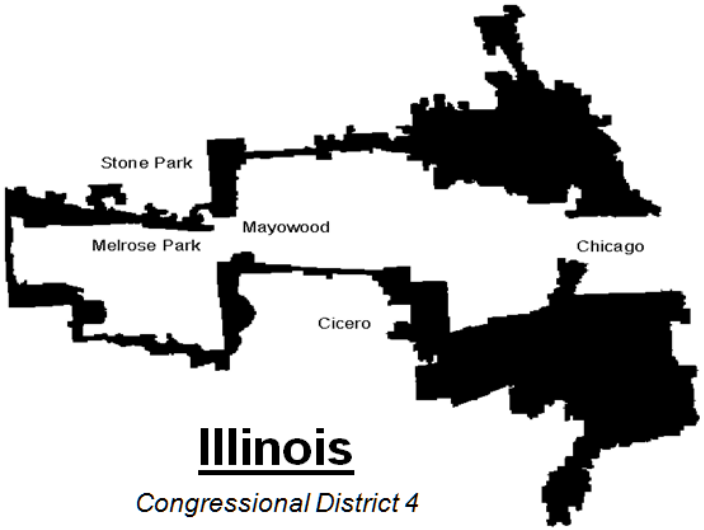
Gerrymandering

Links:

1. A TED-Ed video on gerrymandering
2. A congressional districts map
3. A wikipedia page on gerrymandering
4. A list of congressional districts in the US, with some nice pictures

A 1812 cartoon

A political cartoon by Elkanah Tisdale, published in the Boston Gazette, accompanied the first usage of the word "gerrymander" on March 26, 1812. The term was coined based on a Massachusetts legislature's redrawing of a voting district to favor Governor Elbridge Gerry's reelection; the cartoon satirizes its atypical shape by giving it a dragon's menacing features.



Illinois

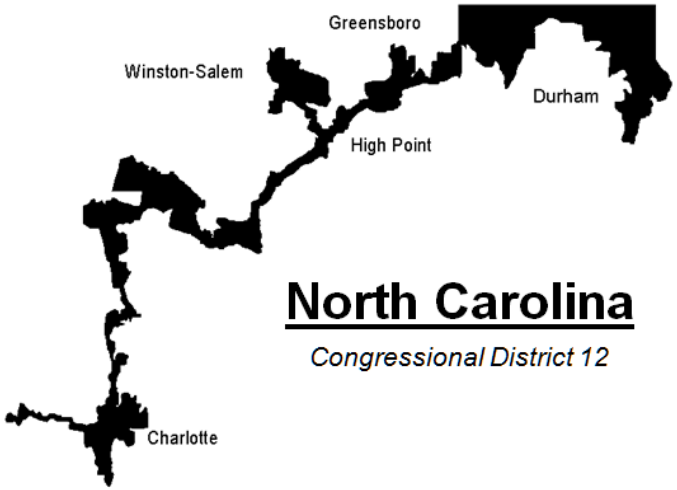
Congressional District 4

Georgia

Congressional District 11

Louisiana

Congressional District 4

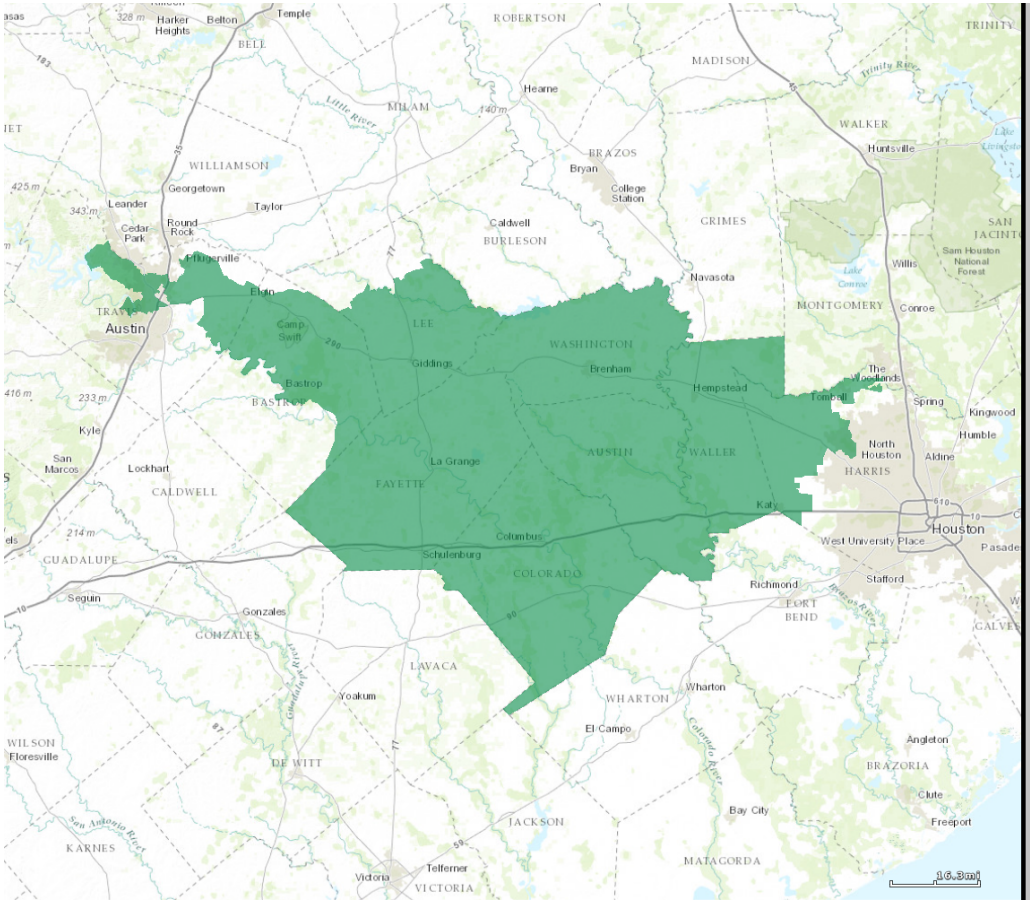


North Carolina

Congressional District 12

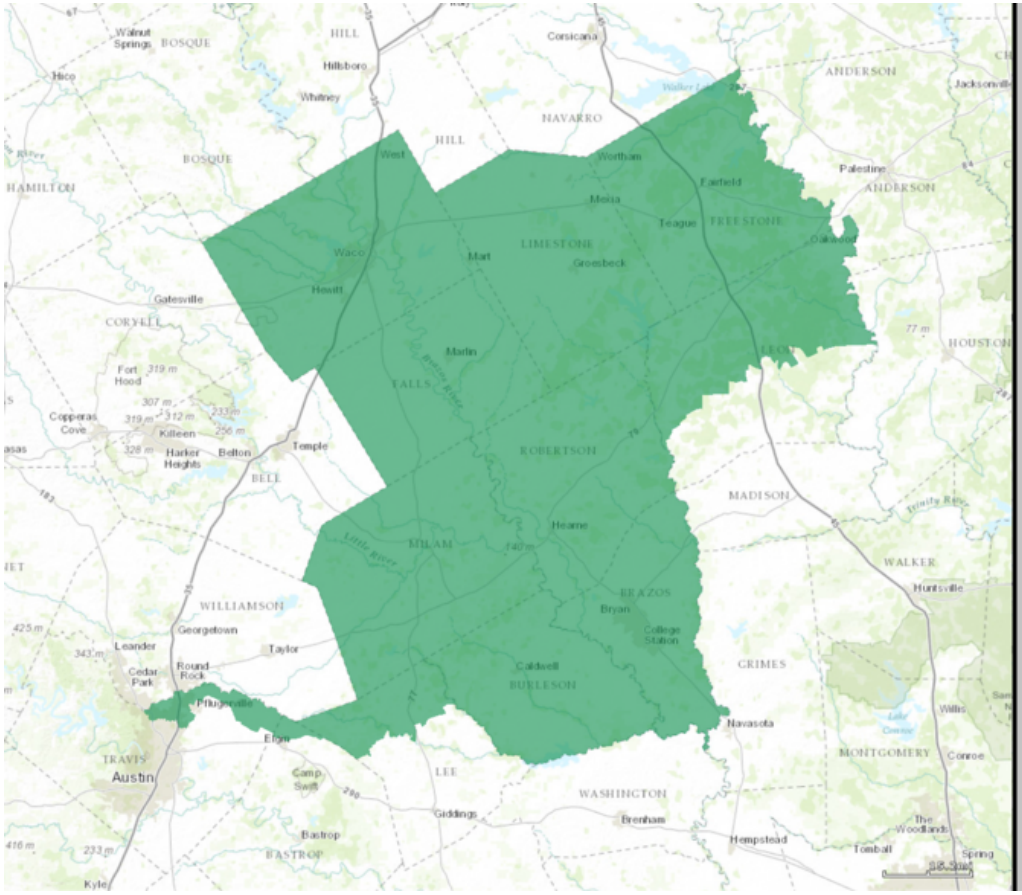
…graphics from 1990s Supreme Court Redistricting Decisions, Peter S. Wattson
www.senate.leg.state.mn.us/departments/scr/REDIST/red907.htm

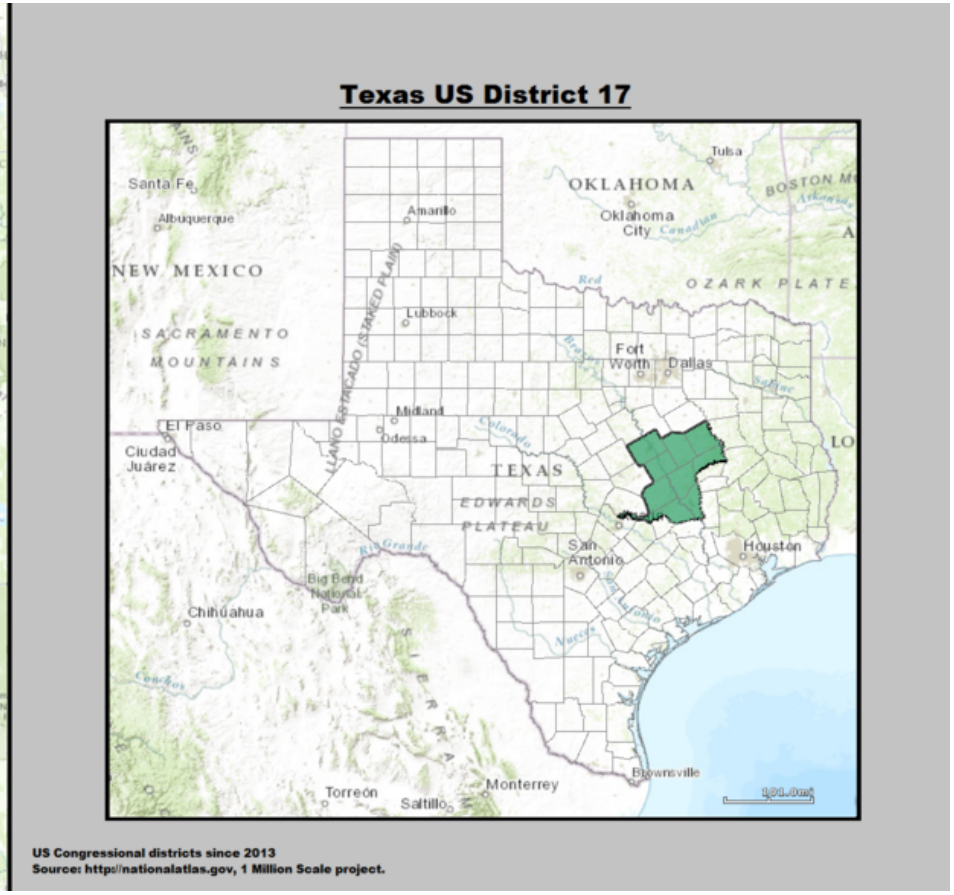
Texas, 10th congressional district



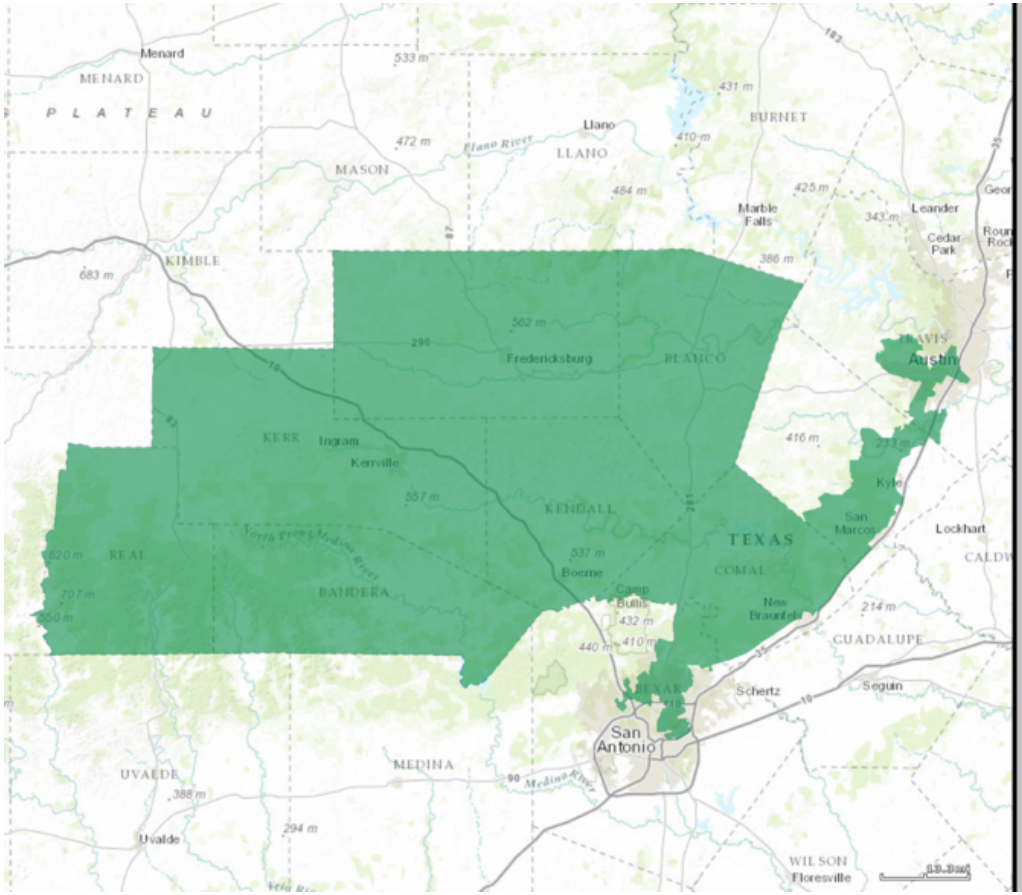
US Congressional districts since 2013
Source: http://nationalatlas.gov, 1 Million Scale project.

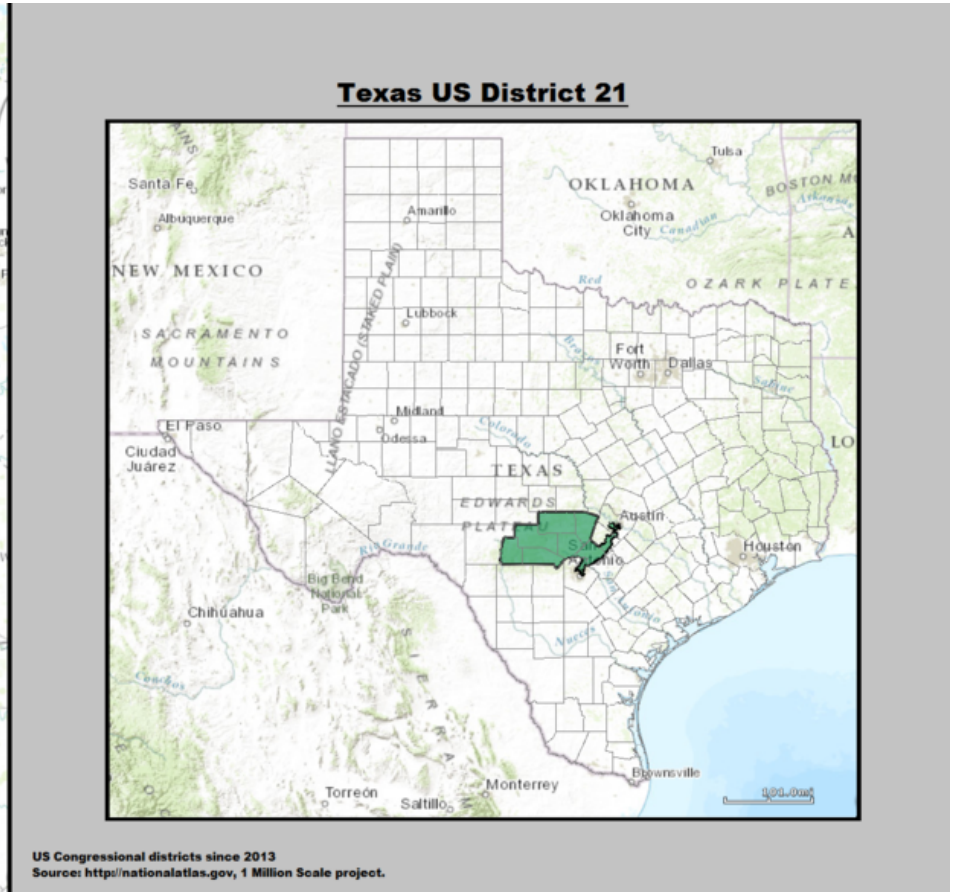
Texas, 17th congressional district





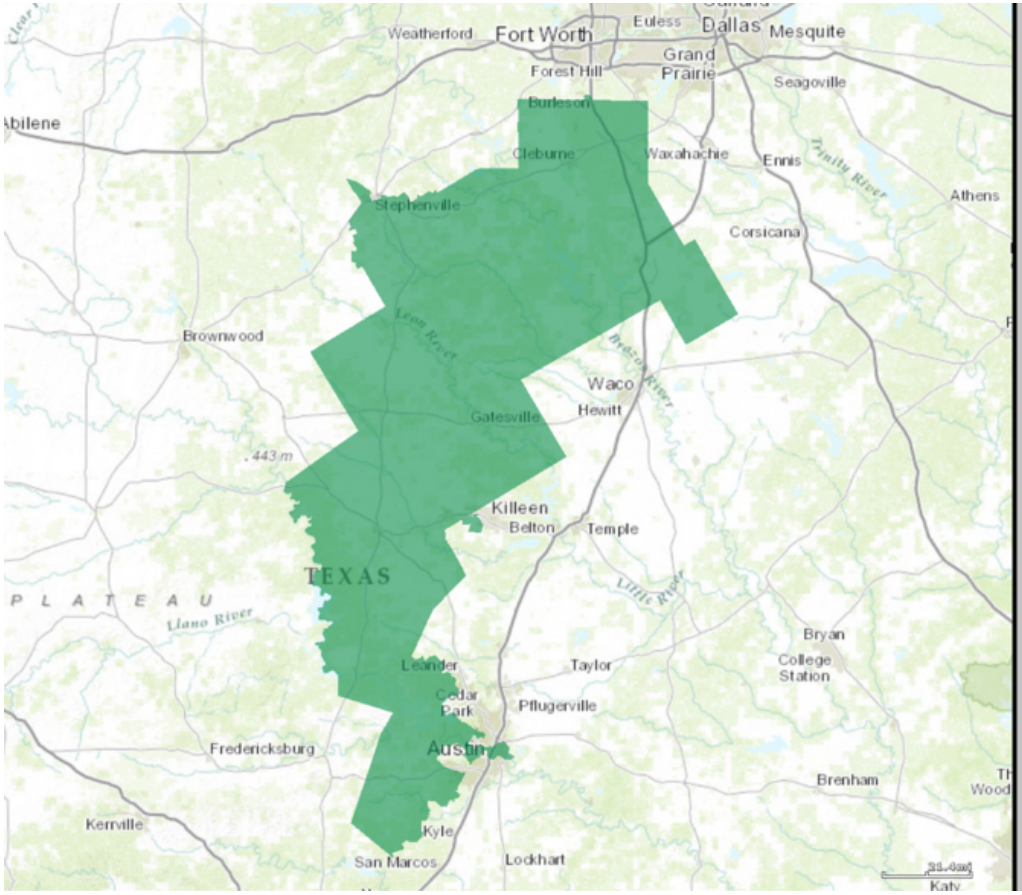
Texas, 21st congressional district

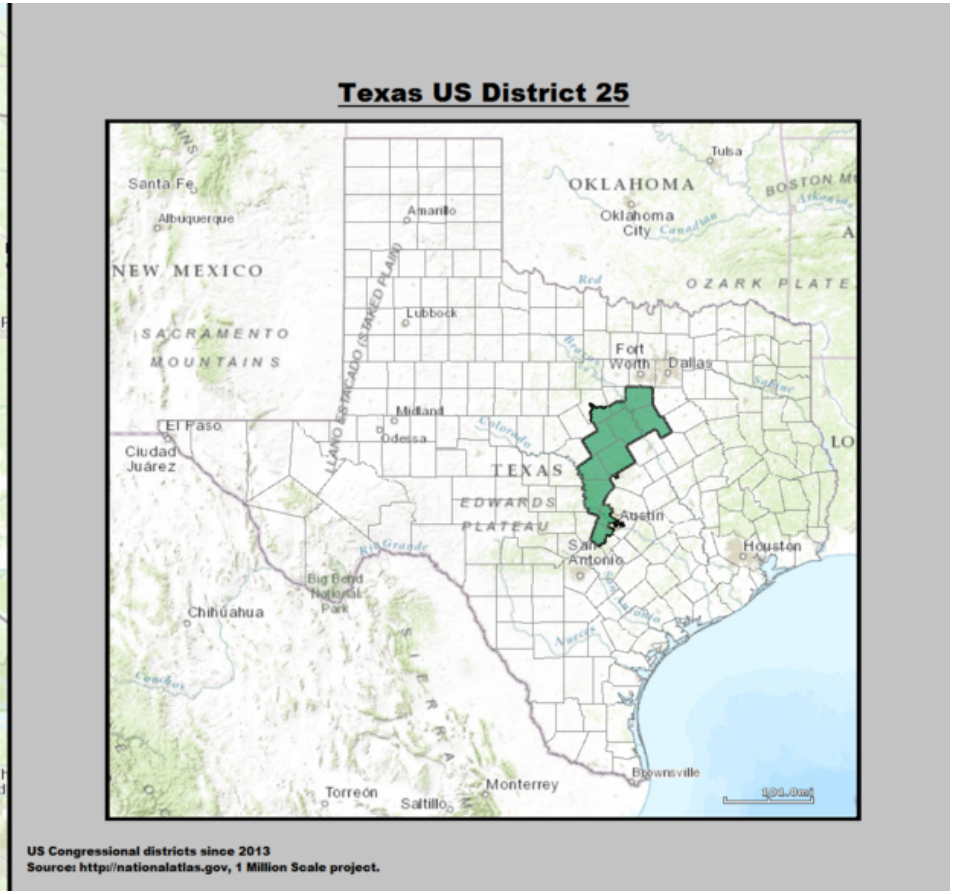




US Congressional districts since 2013
Source: http://nationalatlas.gov, 1 Million Scale project.

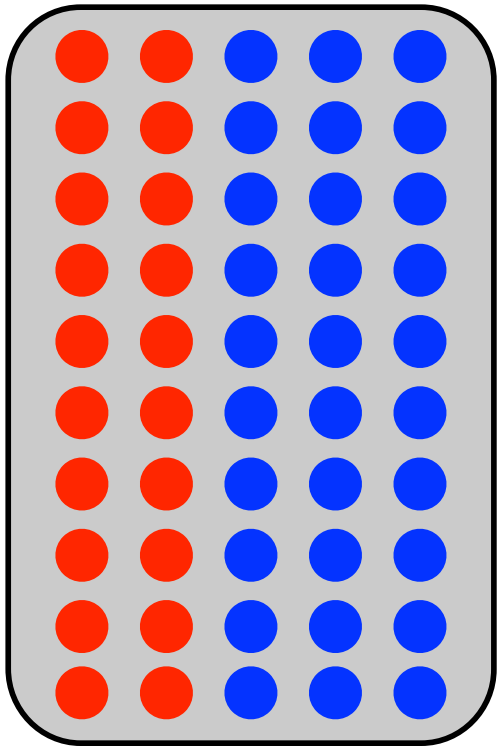
Texas, 25th congressional district





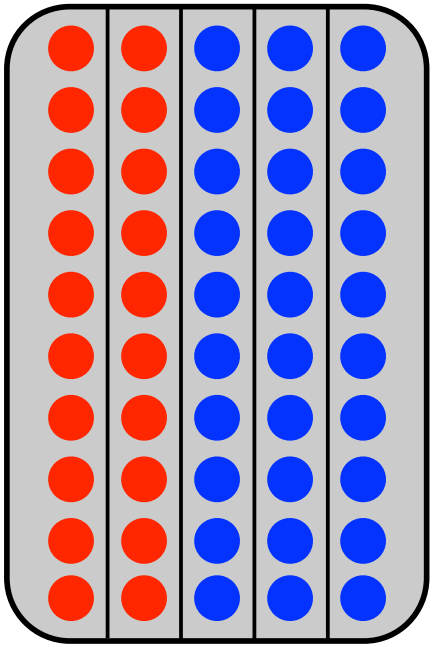
Gerrymandering (cont'd)

- Consider the following "state" (adapted from this Washington Post article), in which there are 50 people, 30 of which are "blue" and 20 of which are "red".

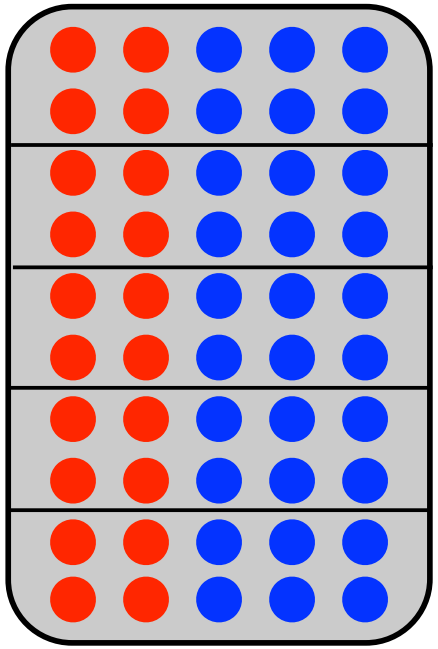


- Can you draw 5 districts, of 10 people each which yields:
  (a) 3 blue, 2 red districts?
  (b) 5 blue districts?
  (c) 2 blue, 3 red districts?

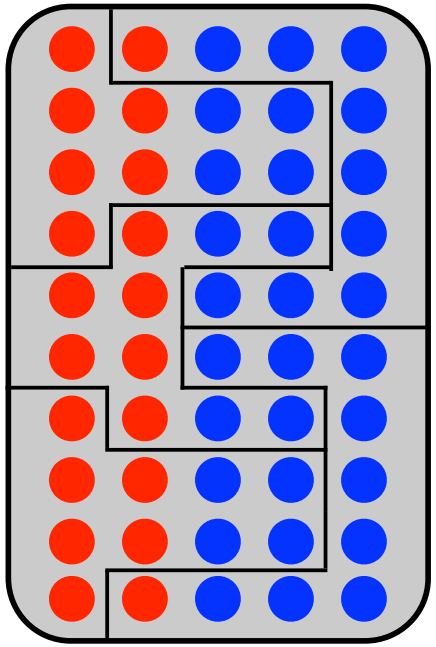
Gerrymandering (cont'd)



3 blue, 2 red



5 blue, 0 red



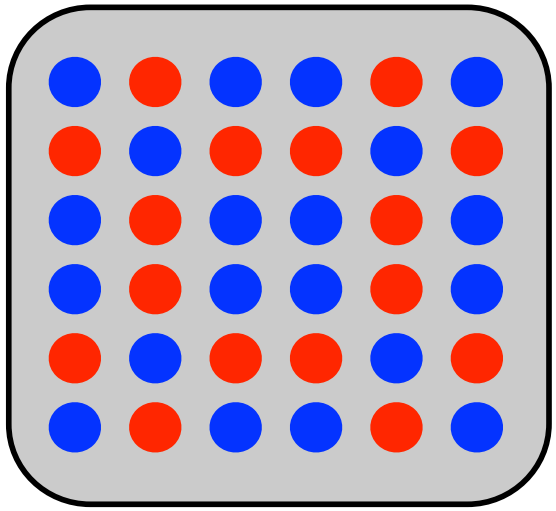
2 blue, 3 red

- The second example demonstrates **cracking**: spreading out your opponents into several districts, diluting their power.

- The last example demonstrates **packing**: placing a large majority of your opponents into a small number of districts which they win easily, but giving you a large number of districts in which you win by a smaller majority.

Gerrymandering (cont'd)

- Consider the following "state" (adapted from this article), in which there are 36 people, 20 of which are "blue" and 16 of which are "red".



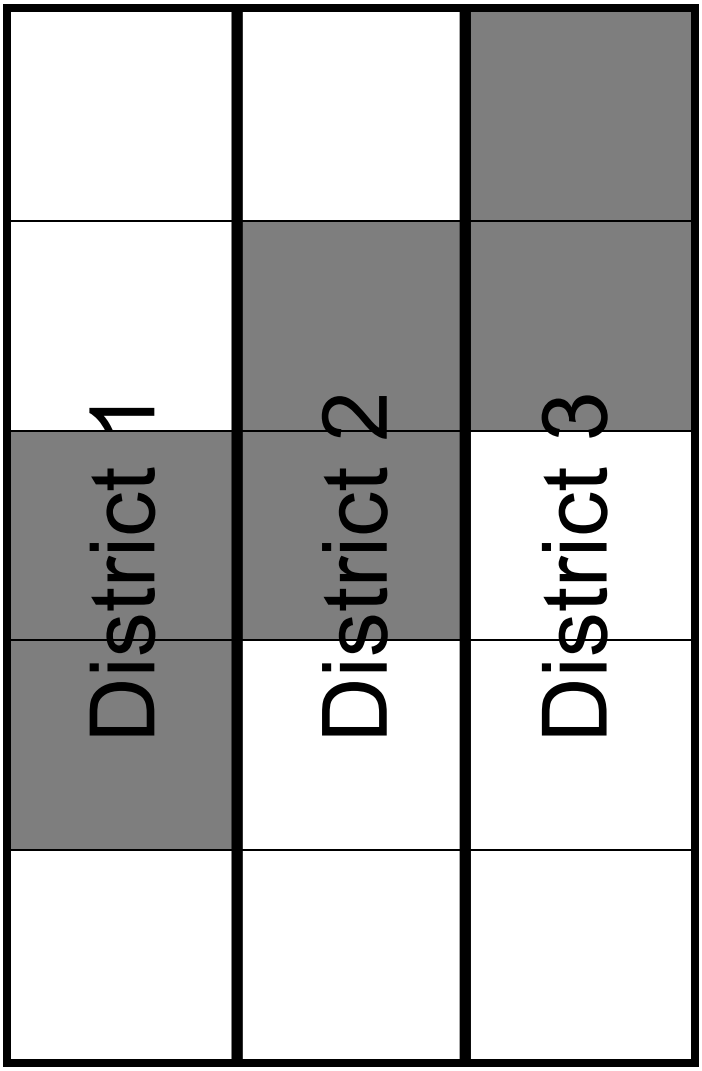
- Can you draw 4 districts, of 9 people each which yields :
  (a) 4 blue districts?
  (b) 1 blue, 3 red districts?

Student Handout 1: Redistricting Maps

Part 1

Map for the State of Squaredom

Below is a map that represents a fictitious state with two political parties, the Gray Party and the White Party. Three Congressional districts were formed with vertical lines marking the district boundaries. In each district, the White Party has a majority of votes. Nice for the White Party, but not an accurate representation of the Gray Party voting population in the state.

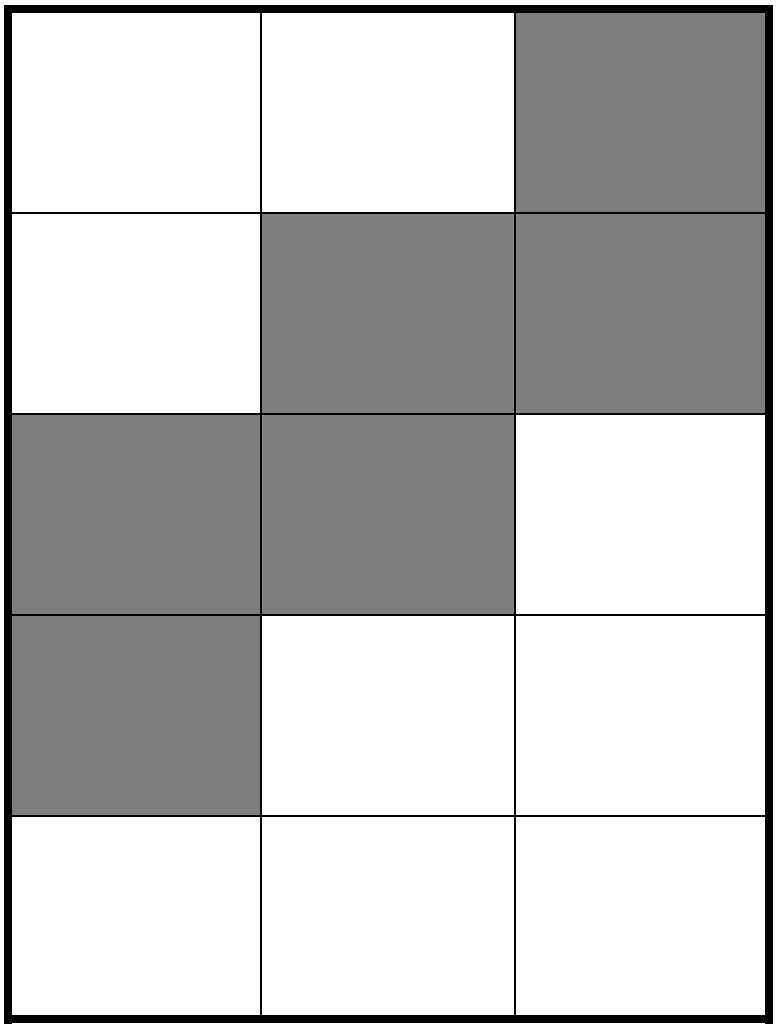


Redistricting Map Worksheet A

Part 2

Redraw the district boundary lines on Map A so that:

- There are three districts
- Each district has five blocks total
- The Gray Party has a majority in one district
- The White Party has majorities in two districts

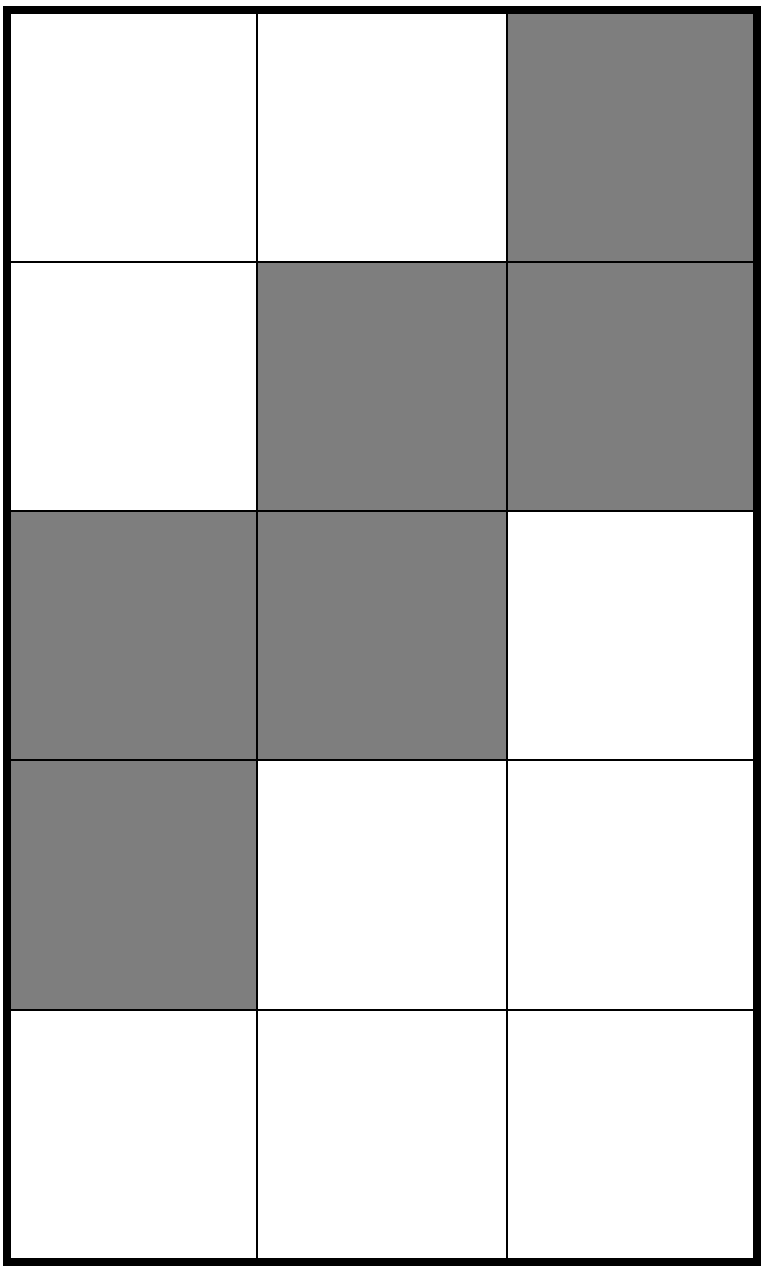


Redistricting Map Worksheet B

Part 3

Redraw the district boundary lines on Map B so that:

- There are three districts
- Each district has five blocks total
- The Gray Party has a majority in two districts
- The White Party has majorities in one district



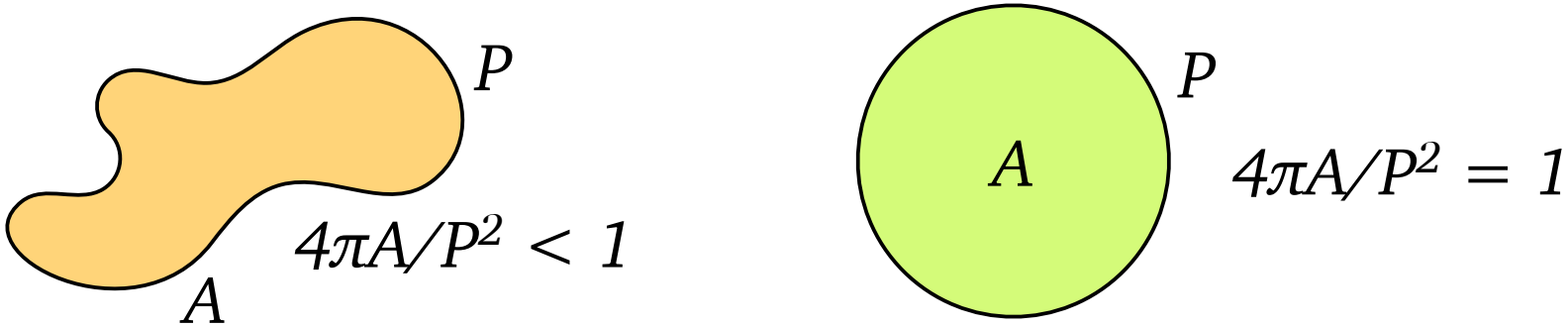
Measures of gerrymandering

- A common description of gerrymandering is “you know it when you see it”.
- But, can we tell *mathematically* if a district is gerrymandered? Such a method would be more objective, and less susceptible to bias.
- There are several competing methods, usually described as **compactness measures**, as they attempt to give a precise meaning to the word “compactness” in the context of congressional districts.
- Each method we present here is, essentially, an answer to the question “how much does the given district differ from an *ideal* district?”
- The methods differ in their understanding of what an “ideal district” is, and how we measure the difference between that and the given district.

The Polsby—Popper ratio

- The first method begins with the assumption that an ideal district should be a circle. To understand it, we need an important result from geometry:

<u>Theorem (The Isoperimetric Inequality):</u> For any "simple closed curve" in the plane with perimeter P (in ft, say), that bounds an area A (in ft^2), we have that $4\pi A \leq P^2$. Equivalently, $4\pi A/P^2 \leq 1$.



- Since $4\pi A = P^2$ when the closed curve is a circle, it follows that amongst all curves with the same perimeter, the circle bounds the maximum possible area.

- The difference between the ratio $4\pi A/P^2$ and 1 is a measure of how much the area enclosed by the curve differs from that of a circle with the *same perimeter*.

The Polsby—Popper ratio (cont'd)

- The **Polsby—Popper ratio** (named for lawyers Daniel Poslby and Robert Popper) for a given congressional district, with perimeter P and area A, is exactly the ratio $4\pi A/P^2$.

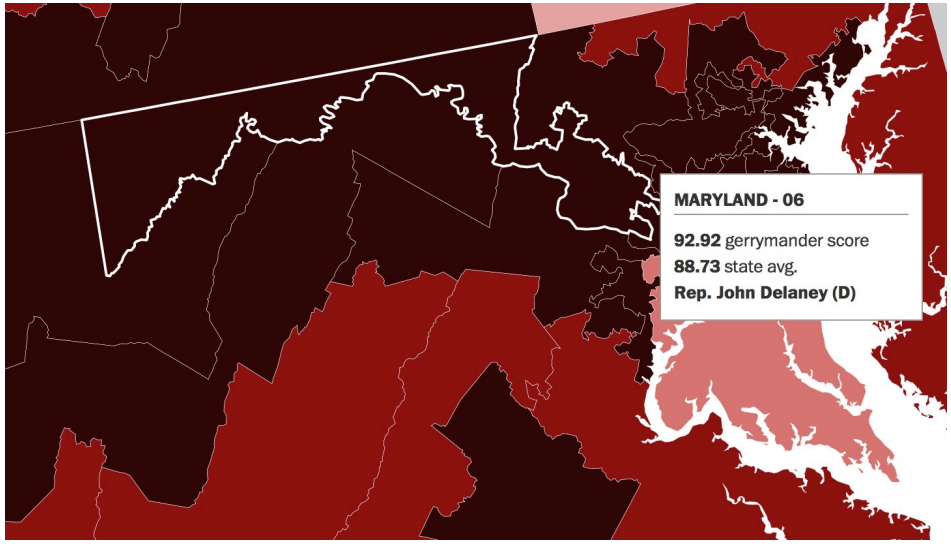
- The intent is that a district with a higher (i.e., closer to 1) ratio is *less* gerrymandered, while one with a lower ratio is *more* gerrymandered.

- A major advantage of this method is that it is extremely easy to determine, using publicly available data.

- In fact, the Washington Post has done it for us (their numbers are the result of multiplying $1\text{-}4\pi A/P^2$ by 100 to obtain an "index"; to obtain the original PP ratios, just divide their "gerrymander score" by 100, and subtract the result from 1).

The Polsby—Popper ratio (cont'd)

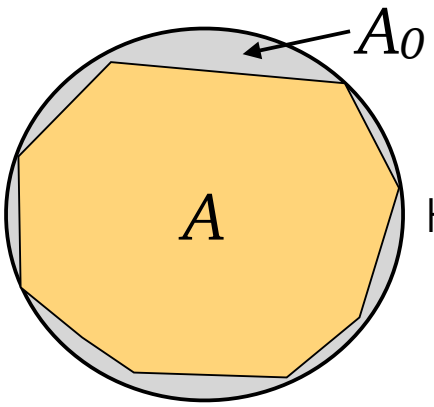
- Can you think of some potential problems with this measure?
  - Squares and rectangles, which don't seem gerrymandered, don't get "perfect" scores.
  - The boundaries of states (which districts must respect), as well as natural boundaries (rivers, lakes, etc) can cause reasonable districts to appear gerrymandered by this measure. For example, consider Maryland's 6th Congressional District:



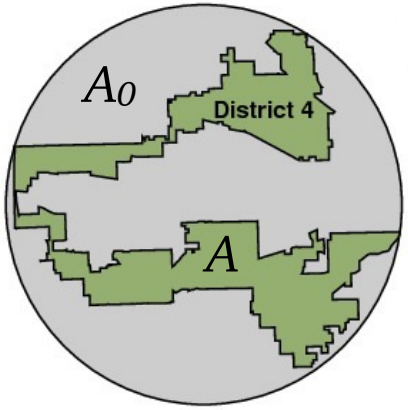
- This district has a very small PP ratio of 0.071.
- But, most of the strange, jagged southern boundary of this district is the Maryland/ West Virginia border, formed by the Potomac River.

The Reock ratio

- This method also begins with the assumption that an ideal district should be a circle, but identifies a different circle as being ideal.
- The **Reock ratio** is the ratio A/A_0, where A is the area of the district, and A_0 is the area of the smallest circle containing the district.
- Again, the intent is that high ratios (i.e., closer to 1) are *less* gerrymandered, while low ratios are *more* gerrymandered.



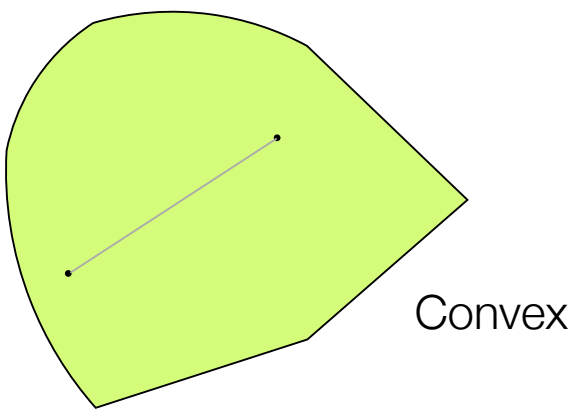
High Reock ratio

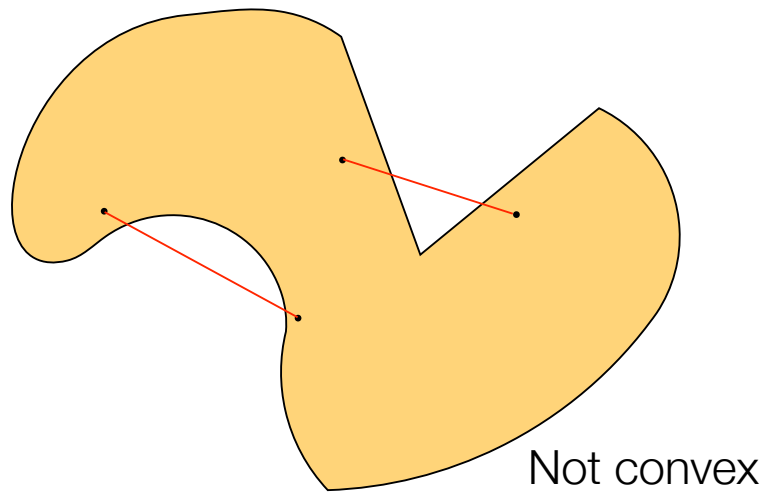


Low Reock ratio
(Illinois 4th district)

The convex hull ratio

- A region is **convex** if whenever two points in the region are connected by a straight line, that line lies entirely within the region.



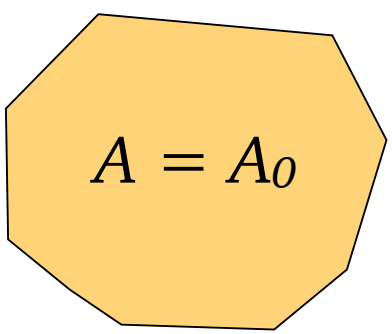


- In particular, both rectangles (and all regular polygons) and circles are convex.

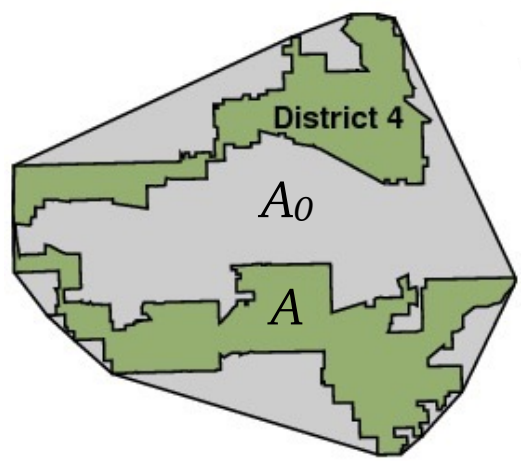
The convex hull ratio (cont'd)

- This method begins with the assumption that an ideal district should be convex.

- The **convex hull ratio** is the ratio A/A_0, where A is the area of the district, and A_0 is the area of the smallest convex region (the **convex hull**) containing the district.



High (perfect!) convex hull ratio

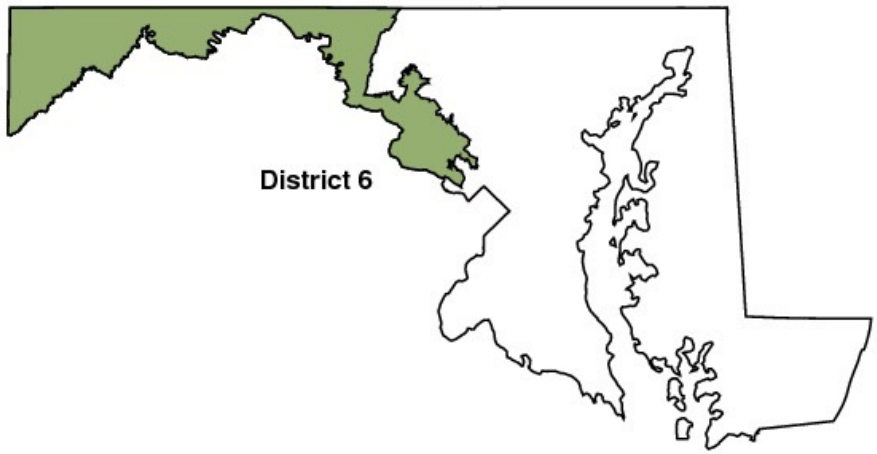


Low convex hull ratio (Illinois 4th district)

Area ratios

- All of the previous measures are examples of **area ratios**, and each is subject to some of the same issues as the Polsby—Popper ratio, namely they do not take into account state and natural boundaries.

- In particular, each gives relatively poor scores to Maryland's 6th Congressional district:
  - Polsby—Popper: 0.071
  - Reock: 0.121
  - Convex hull: 0.562



Bizarreness

- In 2007, economists Christopher Chambers and Alan Miller introduced an alternate measure which addresses some of the difficulties with the area measures we've seen.

- The **bizarreness** of a district is (essentially) the probability that the shortest path *within the state* between two people in the district stays within the district. (For more details, see their paper linked above, or the more elementary explanation in this AMS Feature Column on Congressional Redistricting.)

- While grammatically unfortunate, the intent is that high bizarreness (i.e., close to 1) means a district is less gerrymandered.

- Convex districts still get a measure of 1, but districts whose non-convexity is due to state boarders have bizarreness close to 1 as well.

- In particular, they compute the bizarreness of Maryland's 6th district to be a relatively mild 0.926, but that of Maryland's 3rd district to be an egregious 0.140.