Saturday Morning Math Group – Austin Math Circle

Austin Area Problem Solving Challenge – 2009

Rules

1. The Austin Area Problem Solving Challenge (AAPSC) is a competition for teams of **up to five students each**. Any student who resides in the Austin area and has not yet graduated from high school is eligible. Teams may include students from different schools.

2. This problem set consists of five problems. Each problem is worth a total of 40 points, but a solution that goes above and beyond what is asked in the problem may earn more than 40 points.

3. You may not discuss any of these problems, or any directly related problems, with people other than those on your team until after the problem set is due. However, you may use books, notes, and the Internet to look up mathematical ideas that you think may help with the problems (but not to ask for assistance with the problems themselves). If you need clarification regarding any of the problems, please e-mail Dave Jensen at smmg@math.utexas.edu.

4. While working on the problems, you may use technology (*e.g.* calculators, computers) to perform computations and experiments. However, you must explain how you arrived at each of your solutions; and solutions that can be reproduced and/or obtained without technology will earn higher marks than solutions that require technology.

5. You may submit your solutions in any of the following ways:

- Electronically, to smmg@math.utexas.edu (All electronic submissions must be in .pdf or .doc format)
- By mail, to Saturday Morning Math Group, Department of Math, UT Austin, 1 University Station C1200, Austin, TX 78712
- At the April 19 Math Circle meeting

All solutions must be received by 1 PM, April 19.

Guidelines for Solutions

1. Please fill out the **AAPSC Cover Page** and attach it to the front of your solutions packet. **Do not identify yourself or your teammates in any way in the solutions themselves.**

2. Please write all solutions in complete sentences, and be neat. Partial credit will be awarded for incomplete solutions if these solutions are well-written and contain potentially useful ideas.

3. Solutions may be handwritten or typed. If you wish to submit your solutions electronically, you must use .pdf or .doc format.

4. Please send all of your solutions in one submission; do not send several separate packets (or send part of your work by mail and the rest of it electronically).

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Cover Sheet

	Name	**E-mail address**
Team		
Team Member #1		
Team Member #2		
Team Member #3		
Team Member #4		
Team Member #5		

Notes:

1. Please come up with a team name that is unique; your team name may include the name of your school (if all team members are from the same school), but it should also contain something that distinguishes you from other teams from your school that might submit solutions.
2. Please include a working e-mail address for each team member. If your team wins a prize, we will need to contact you and ask you which of several available prizes you want.
3. If you send your solutions via e-mail, you do not need to include this Cover Sheet; instead, please provide all of the requested information in the body of your e-mail, and include the solutions as attachments.

Good luck, and enjoy the problems!

Problem 1: The Extra Credit Problem

Mr. Patterson, an eccentric math teacher, has an unusual way of awarding extra credit in his classes. For a given grading period, each student is given a certain nonzero number of *required* grades, and a certain nonzero number of *optional* (or "extra credit") grades. At the end of the grading period, a student receives a report containing both the required and optional grades, with the optional grades in brackets. A student's grade report might then look like the following:

86 79 92 85 80 77 [75] [99] [88]

Mr. Patterson then allows the student to drop whichever optional grade(s) she wishes, if she wishes to drop any at all. All of the remaining grades are then averaged to give the student's grade for the grading period. It is the student's responsibility to choose which grades to drop so that she gets the highest grade possible.

So in this example, the student could choose to drop the 75 and the 88; her grade would then be

$$\text{Grade} = \frac{86+79+92+85+80+77+99}{7} \approx 85.4$$

She could also choose not to drop any of her optional grades; if she did this, her grade would be

$$\text{Grade} = \frac{86+79+92+85+80+77+75+99+88}{9} \approx 84.6$$

In this problem, we'll examine some different ways in which students in Mr. Patterson's class decide which optional grades to keep and which ones to drop, and determine whether they give the best possible averages. In the following problems, unless otherwise specified, a grade report may contain any nonzero number of required grades, and any nonzero number of optional grades.

(a) Suppose that Dave receives the following grade report:

84 76 97 93 88 [74] [93] [85] [96] [94]

If Dave doesn't include any of his optional grades in his average, his grade will be 87.6, a B-plus. Is there a combination of optional grades Dave can include in his average so that he will get at least an A-minus (90 or above)?

(b) Suppose that in each grading period, Dave only cares about whether he can get an average of at least 90; if he can get an average of at least 90, he doesn't care about getting the highest average possible. Prove that if it is possible for him to get an average of at least 90, then he can do it by keeping only those optional grades that are greater than or equal to 90.

(c) Whenever Erica gets her grade report at the end of the grading period, she calculates the average of her required grades, and calls this average x. She then chooses to keep all of her optional grades that are greater than or equal to x, and drops the ones that are less than x. Prove or disprove that Erica's method will always result in the highest possible average.

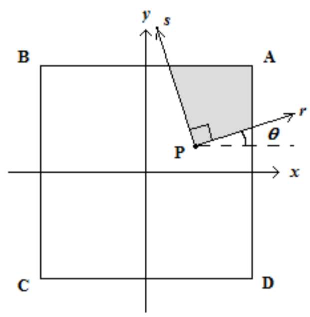
(d) Whenever Franco gets his grade report at the end of the grading period, he arranges all of his optional grades from best to worst, and calls them $y_1, y_2, y_3, \ldots, y_n$, with $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n$. He then defines k to be the least positive integer such that y_k is less than the average of his required grades and his $k-1$ highest optional grades. (If there is no such positive integer, he defines k to be $n+1$.) He then chooses to keep his $k-1$ highest optional grades, and drop the rest. Prove or disprove that Franco's method will always result in the highest possible average.

Problem 2: The Right Stuff

In this problem, the square $ABCD$ will always be the square in the coordinate plane with vertices $A = (1, 1)$, $B = (-1, 1)$, $C = (-1, -1)$, and $D = (1, -1)$. Suppose we choose a point $P = (a, b)$, with $-1 \le a \le 1$ and $-1 \le b \le 1$, and choose an angle θ. We then draw two rays $\vec{r}$ and $\vec{s}$ from point P, defined by

$$\vec{r} = \{(a + t\cos\theta, b + t\sin\theta) : t \ge 0\} \quad \text{and} \quad \vec{s} = \{(a - t\sin\theta, b + t\cos\theta) : t \ge 0\}.$$

In other words, we obtain the ray $\vec{r}$ by starting with the ray from point P that points rightward, and rotating this ray counterclockwise through an angle of θ about the point P. We then obtain $\vec{s}$ by rotating the ray $\vec{r}$ counterclockwise through an angle of 90° about the point P. We define R to be the region consisting of all points inside or on the square $ABCD$ that also lie inside or on the right angle formed by rays $\vec{r}$ and $\vec{s}$. (This is the shaded region below.)



(a) Prove that if $P = (0, 0)$, then the area of R is independent of θ.

(b) Suppose we are given that the ray $\vec{r}$ intersects the side AD, and the ray $\vec{s}$ intersects the side AB. In terms of a, b, and θ, what is the area of region R? Prove your answer.

(c) If the ray $\vec{r}$ intersects the side AD at a point other than A and D, and the ray $\vec{s}$ intersects the side AB at a point other than A and B, then the region R is a quadrilateral. Find, with proof, the radius of the smallest circle that contains the region R. (Your answer should be in terms of a, b, and θ.)

(d) Under what conditions (on a, b, and θ) is it true that the ray $\vec{r}$ intersects the side AD and the ray $\vec{s}$ intersects the side AB? Your answer should be a theorem of the form "The rays $\vec{r}$ and $\vec{s}$ intersect the sides AD and AB, respectively, if and only if ..." Prove your theorem.

(e) Suppose we are given that the ray $\vec{r}$ intersects the side AD, and the ray $\vec{s}$ intersects the side BC. In terms of a, b, and θ, what is the area of region R? Prove your answer.

(f) Suppose that $P = (-\frac{1}{2}, -\frac{1}{2})$, and that rays $\vec{r}$ and $\vec{s}$ meet sides AD and BC, respectively, at $(1, \frac{1}{4})$ and $(-1, \frac{1}{2})$. What is the radius of the smallest circle that contains the region R? Prove your answer.

(g) Give and prove a formula that will determine, for any choice of a, b, and θ, the area of R. You may need to use different formulas for different choices of a, b, and θ; if so, be sure to explain under what circumstances each formula applies. (You are free to make any assumption about a, b, and θ that doesn't significantly alter the generality of your formula; for example, you may assume by rotating the entire picture about the origin that $\vec{r}$ intersects the side AD.)

Problem 3: Warning: May Contain MSG

The *Monotone Subsequence Game* (or MSG) is a game for two players (whom we'll call Alice and Bob, with Alice always moving first). The game begins with a sequence of distinct positive integers, such as the following:

$$3 \quad 8 \quad 1 \quad 6 \quad 4 \quad 2 \quad 7 \quad 5$$

The two players take turns crossing out numbers on the list. A player may cross out as many numbers as he/she wants, as long as the numbers form a *monotone subsequence* – that is, either an increasing subsequence or a decreasing subsequence. The numbers crossed out do not have to appear consecutively in the list, nor do they have to be consecutive integers. So for example, on her first turn, Alice may cross out the increasing subsequence $(3, 6, 7)$, or she may cross out the decreasing subsequence $(8, 6, 4, 2)$. She may also choose to cross out only one number, since any one number by itself is considered both an increasing subsequence and a decreasing subsequence. On each turn, a player *must* cross out at least one number. A player cannot cross out any numbers that have already been crossed out. The player who crosses out the last number (possibly as part of a longer subsequence) wins the game. For a given game of MSG (that is, a given starting sequence), we say that a player (either Alice or Bob) has a *winning strategy* if it is possible for that player to win the game no matter what the other player does.

(a) Suppose that Alice and Bob play an MSG starting with a sequence containing only the integers from 1 to 5. Prove that Alice has the winning strategy, regardless of the order of the sequence.

(b) Show that there is an MSG starting with the integers 1 through 6 for which Bob has the winning strategy.

(c) Suppose that Alice and Bob play an MSG starting with the sequence $(3, 8, 1, 6, 4, 2, 7, 5)$ shown above. Which player has the winning strategy? Prove your answer.

(d) Suppose that Alice and Bob play an MSG starting with the sequence

$$n+1 \quad n+2 \quad n+3 \quad \cdots \quad 2n-1 \quad 2n \quad 1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n$$

where n is a positive integer. Determine, with proof, which player has the winning strategy. (Your answer may depend on n.)

(e) Suppose that Alice and Bob play an MSG starting with the integers 1 through 5. Prove that Alice can always cross out at least three integers on her first turn (though it may not be advantageous for her to do so).

(f) Suppose that Alice and Bob play an MSG starting with the integers 1 through 14. Prove that Alice can always cross out at least four integers on her first turn (though it may not be advantageous for her to do so).

Problem 4: Divisorful!

In this problem, we say that a set S of positive integers is *divisorful* if and only if it has the following property: if n is in the set S, then every positive integer divisor of n is also in S. So for example, the set $\{1, 2, 3, 4, 6, 8, 16\}$ is divisorful, but the set $\{1, 4, 16, 64\}$ is not divisorful because 4 is in the set, but 2, a divisor of 4, is not in the set.

(a) Suppose that T is a set of positive integers. Then the *divisorful set generated by T*, denoted $\langle T \rangle$, is defined to be the smallest divisorful set that has contains all of the integers in T. For each of the following sets T, determine how many integers are in $\langle T \rangle$:

1. $T = \{6, 15, 25\}$
2. $T = \{30, 40, 50\}$
3. $T = \{2^{17} \cdot 3^{76}, 2^{18} \cdot 5^{65}\}$
4. $T = \{2^{39}, 2^{29} \cdot 3^{29}, 2^{19} \cdot 3^{19} \cdot 5^{19}, 2^9 \cdot 3^9 \cdot 5^9 \cdot 7^9\}$

(b) Suppose that S is a divisorful set of positive integers. Then we say that A is a *generating set for S* if $\langle A \rangle = S$. Suppose that n is a positive integer. Prove that the set S containing all of the positive integers from 1 to n is divisorful, and find a generating set for S that is as small as possible.

(c) Suppose that S is a divisorful set of positive integers, and suppose that A is a generating set for S. We say that A is a *minimal generating set for S* if, for any proper subset $B \subset A$, $\langle B \rangle \neq S$. (In other words, A is minimal if there is no strictly smaller subset of A that generates S.) Let S be the set

$$S = \{2^a \cdot 3^b \cdot 5^c : a \geq 0,\ b \geq 0,\ c \geq 0,\ a + b + c \leq 10\}.$$

Prove that S is divisorful, and find, with proof, a minimal generating set for S.

(d) Prove that every finite divisorful set has a unique minimal generating set. (Here, by "unique" we mean that there is only one minimal generating set.)

(e) For each of the following, either give (with proof) an example of a divisorful set with the properties indicated, or prove that no such example exists.

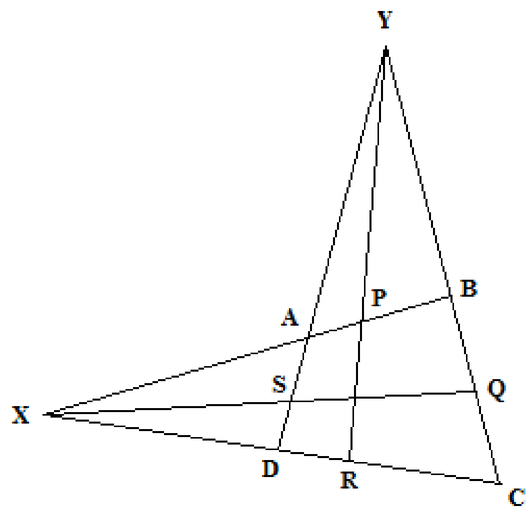
1. An infinite divisorful set that has a unique minimal generating set.
2. An infinite divisorful set that has more than one minimal generating set.
3. An infinite divisorful set that does not have a minimal generating set.

(f) Does there exist an infinite divisorful set that contains exactly one integer from each interval of the form $\{10n + 1, 10n + 2, 10n + 3, \ldots, 10n + 10\}$, where n is a nonnegative integer? (That is, is there a divisorful set that contains exactly one of the integers from 1 to 10, exactly one of the integers from 11 to 20, exactly one of the integers from 21 to 30, and so on?) Prove your answer.

(g) The set of positive integers from 1 to 50 has ${}_{50}C_7$ – almost 100 million – seven-integer subsets. How many of these subsets are divisorful? Show how you arrived at your answer.

Problem 5: Midline Crisis

In this problem, $ABCD$ is a quadrilateral with no two sides parallel. Points P, Q, R, and S are located on sides AB, BC, CD, and DA, respectively, so that the lines AB, QS, and CD all meet at the point X, and the lines AD, PR, and BC all meet at the point Y. The lines intersect so that A is between B and X on the line AB, and A is between D and Y on the line AD.



(a) Prove or disprove that the angle $\angle BAD$ is obtuse.

(b) Prove or disprove that the angle $\angle XAY$ is greater than each of the angles $\angle XBY$, $\angle XCY$, and $\angle XDY$.

(c) Prove that

$$\frac{XA}{AS} \cdot \frac{XC}{CQ} = \frac{XB}{BQ} \cdot \frac{XD}{DS}.$$

(d) Prove that

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$