# Saturday Morning Math Group – Austin Math Circle

## Austin Area Problem Solving Challenge – 2010

### Rules

**1.** The Austin Area Problem Solving Challenge (AAPSC) is a competition for teams of **up to five students each.** Any student who resides in the Austin area and has not yet graduated from high school is eligible. Teams may include students from different schools.

**2.** This problem set consists of five problems. Each problem is worth a total of 40 points, but a solution that goes above and beyond what is asked in the problem may earn more than 40 points.

**3.** You may not discuss any of these problems, or any directly related problems, with people other than those on your team until after the problem set is due. However, you may use books, notes, and the Internet to look up mathematical ideas that you think may help with the problems (but not to ask for assistance with the problems themselves). If you need clarification regarding any of the problems, please e-mail Neil Hoffman at smmg@math.utexas.edu.

**4.** While working on the problems, you may use technology (*e.g.*, calculators, computers) to perform computations and experiments. However, you must explain how you arrived at each of your solutions; and solutions that can be reproduced and/or obtained without technology will earn higher marks than solutions that require technology.

**5.** You may submit your solutions in any of the following ways:

- Electronically, to smmg@math.utexas.edu (All electronic submissions must be in .pdf or .doc format)
- By mail, to Saturday Morning Math Group, Department of Math, UT Austin, 1 University Station C1200, Austin, TX 78712
- At the April 11 Math Circle meeting

All solutions must be received by **1 PM, April 11**.

### Rules

**1.** Please fill out the **AAPSC Cover Sheet** and attach it to the front of your solutions packet. **Do not identify yourself or your teammates in any way in the solutions themselves.**

**2.** Please write all solutions in complete sentences, and be neat. Partial credit will be awarded for incomplete solutions if these solutions are well-written and contain potentially useful ideas.

**3.** Solutions may be handwritten or typed. If you wish to submit your solutions electronically, you must use .pdf or .doc format.

**4.** Please send all of your solutions in one submission; do not send several separate packets (or send part of your work by mail and the rest of it electronically).

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**Cover Sheet**

| | **Name** | **E-mail address** |
|---|---|---|
| **Team** | | |
| **Team Member #1** | | |
| **Team Member #2** | | |
| **Team Member #3** | | |
| **Team Member #4** | | |
| **Team Member #5** | | |

**Notes:**

1. Please come up with a team name that is unique; your team name may include the name of your school (if all team members are from the same school), but it should also contain something that distinguishes you from other teams from your school that might submit solutions.

2. Please include a working e-mail address for each team member. If your team wins a prize, we will need to contact you and ask you which of several available prizes you want.

3. If you send your solutions via e-mail, you do not need to include this Cover Sheet; instead, please provide all of the requested information in the body of your e-mail, and include the solutions as attachments.

**Good luck, and enjoy the problems!**

**Problem 1: This Almost Magic Moment**

You are probably already familiar with the oft-celebrated number array called the *magic square*: a $n$-by-$n$ array, containing each of the positive integers from 1 to $n^2$ exactly once, in which the sum of the numbers on each row and the sum of the numbers on each column are the same number, called the *magic sum*. (Traditionally, the numbers on each main diagonal are also required to add up to the magic sum; however, for purposes of this problem, we will not impose any conditions on the diagonals of the square.) One example of a 3-by-3 magic square is shown below:

| | | |
|---|---|---|
| 1 | 6 | 8 |
| 5 | 7 | 3 |
| 9 | 2 | 4 |

In this square, the magic sum is 15; the sum of the numbers on each row is 15, as is the sum of the numbers on each column.

**(a)** A *magic rectangle* is an $m$-by-$n$ array, containing each of the positive integers from 1 to $mn$ exactly once, in which every row sum and every column sum is the same number, called the *magic sum*. Prove that there are no magic rectangles with $m \neq n$; that is, the only magic rectangles are magic squares.

If the purpose of this problem were to explore the world of magic rectangles, this would be a rather boring problem. So let us instead take a look at the magic square's lesser-known (but equally interesting!) cousins: almost magic rectangles. An *almost magic rectangle* is an $m$-by-$n$ array, containing each of the positive integers from 1 to $mn$ exactly once, in which every row sum and every column sum is one of two numbers, which we will call the *almost magic sums*. We will use the symbol $A_{m,n}(S,T)$, where $S < T$, to denote an $m$-by-$n$ almost magic rectangle with almost magic sums $S$ and $T$; that is, an $m$-by-$n$ array, containing the integers from 1 to $mn$, in which each row sum is either $S$ or $T$, and each column sum is either $S$ or $T$. (Note that we are allowing for the possibility that $m = n$; an almost magic rectangle with $m = n$ is called an *almost magic square*.)

**(b)** Is it possible to construct a 2-by-2 almost magic square? If so, find all ordered pairs $(S,T)$ such that there exists an $A_{2,2}(S,T)$. If not, prove that no such almost magic squares exist.

**(c)** Prove that there is only one ordered pair $(S,T)$ such that there exists an $A_{2,3}(S,T)$, and construct such an almost magic rectangle.

**(d)** Suppose that $n$ is an odd number such that $n > 3$. Prove that there are no 2-by-$n$ almost magic rectangles.

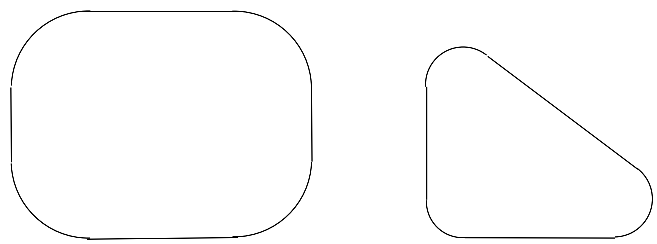
**(e)** For each of the following, either give an example of an almost magic rectangle of the type given, or prove that no such almost magic rectangle exists.

**i.** An $A_{3,3}(14,16)$ **ii.** An $A_{3,3}(14,17)$ **iii.** An $A_{3,7}(21,110)$

**(f)** Find, with proof, all ordered pairs $(S,T)$ of positive integers, with $S < T$, such that there exists an $A_{4,5}(S,T)$. For any almost magic rectangles you construct in this part of the problem, discuss how you obtained the rectangles: whether you used technology, or whether you developed a method that can be carried out by hand. Also discuss whether your method for generating magic rectangles can be generalized to other cases (*i.e.*, other dimensions and other almost magic sums).

**Problem 2: Rounding Off**

Suppose that we take a convex polygon $P$, choose a radius $r$, and replace each corner of the polygon with a circular arc of radius $r$ such that the sides of the polygon that form that corner, if continued, would be tangent to the circular arc. The figure obtained by performing this operation on each corner of $P$ will be called the *$r$-rounding of $P$*. As examples, the 1-rounding of a 3-by-4 rectangle and the 1/2-rounding of a 3-4-5 right triangle are shown below:



For purposes of this problem, we will assume that $r$ is always small enough so that the circular arcs on two adjacent corners do not intersect; that is, so that there is a straight line segment remaining on each side.

**(a)** Compute the area, perimeter, and diameter of the 1-rounding of a 3-by-4 rectangle. Answers alone will suffice; you will be asked for the proof of a similar calculation in part (c). (*Note:* The *diameter* of a convex figure in the plane is defined to be the length of the longest line segment that can be drawn inside the figure, with both of its endpoints on the boundary of the figure.)

**(b)** Let $a$ and $b$ be positive numbers. For what values of $r$ can we construct the $r$-rounding of an $a$-by-$b$ rectangle? (Your answer should be in terms of $a$ and $b$.)

**(c)** Suppose we construct the $r$-rounding of an $a$-by-$b$ rectangle. In terms of $a$, $b$, and $r$, find, with proof, the area, perimeter, and diameter of this figure.

**(d)** Compute the area, perimeter, and diameter of the 1/2-rounding of a 3-4-5 right triangle. Answers alone will suffice; you will be asked for the proof of a similar calculation in part (f).

**(e)** Let $a$, $b$, and $c$ be positive numbers. For what values of $r$ can we construct the $r$-rounding of a triangle with sides of length $a$, $b$, and $c$? (Your answer should be in terms of $a$, $b$, and $c$.)

**(f)** Suppose we construct the $r$-rounding of a triangle with sides of length $a$, $b$, and $c$. In terms of $a$, $b$, $c$, and $r$, find, with proof, the area and perimeter of this figure.

**Problem 3: A Gorgon-tuan Counting Problem**

King Neil rules over his land from the comfort of a small castle in the hills. In front of his castle is a network of walkways; these walkways, together with their endpoints, form a structure known to mathematicians as a *graph.* A *graph* is a collection of vertices, with some pairs of vertices joined by edges. Some examples are shown below:

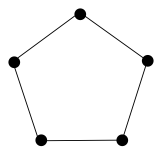


Figure 1

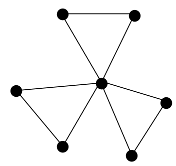


Figure 2

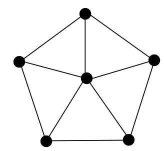


Figure 3

In order to protect his castle from unwanted visitors, Neil posts a gorgon at each vertex of the graph. A gorgon is a mythical creature whose gaze can turn any creature into stone, but only if that creature looks into the gorgon's eyes. Each gorgon is instructed to fix its gaze along one of the edges touching the vertex it is standing on. (So in the example in Figure 1 above, Neil posts five gorgons, with each gorgon instructed to stare down one of two adjacent edges.) The problem is that if two gorgons on opposite ends of the same edge stare at each other, they will both turn to stone, rendering them useless. Therefore, Neil must make sure that no two gorgons on adjacent vertices are facing each other.

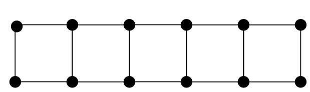
**(a)** A *connected graph* is a graph in which it is possible to travel from any vertex to any other vertex by traveling along a sequence of consecutive vertices and edges. In a graph, a *cycle* is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$, where $e_i$ is an edge joining $v_{i-1}$ and $v_i$ for each $i \leq k$, and all of the vertices and edges are distinct, except that $v_0 = v_k$. A *tree* is a connected graph that has no cycles. Prove that if Neil's network of walkways is a tree, then it is impossible for him to orient his gorgons so that no gorgons turn to stone.

**(b)** Suppose that Neil's network of walkways is a cycle with $n$ vertices. (The graph shown in Figure 1 above is a cycle with five vertices.) How many ways are there for Neil to orient his gorgons so that no gorgons turn to stone?

**(c)** A *rose with $n$ $k$-petals* is a graph consisting of $n$ cycles (called petals), each on $k$ vertices. There is one vertex, called the *center*, that is a vertex of every petal. (The rose with three 3-petals is shown in Figure 2 above.) If Neil's network of walkways is a rose with $n$ $k$-petals, how many ways are there for Neil to orient his gorgons so that no gorgons turn to stone?

**(d)** A *wheel with $n$ spokes* is a graph with $n+1$ vertices. One vertex is the *hub* of the wheel and is adjacent to each of the other $n$ vertices. The other $n$ vertices form a cycle with $n$ vertices. (The wheel with five spokes is shown in Figure 3 above.) If Neil's network of walkways is a wheel with $n$ spokes, how many ways are there for Neil to orient his gorgons so that no gorgons turn to stone?

**(e)** How many ways are there for Neil to orient gorgons on the vertices of the grid below so that no gorgons turn to stone? (You are encouraged to find a way to answer this question without using technology or brute-force counting, though you are free to use technology to check your answer.)



**Problem 4: Gotta Keep 'Em Separated**

In this problem, we will explore the probability that a line segment in a figure, chosen in some random way, separates a pair of points in the figure that is also chosen in some random way.

**(a)** Consider the square in the coordinate plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. Suppose we choose a number $k$ at random from the interval $[0, 1]$, and draw the line segment from $(k, 0)$ to $(k, 1)$. We also choose two points $A$ and $B$ at random in the interior of the given unit square. What is the probability that $A$ and $B$ lie on opposite sides of the randomly-chosen vertical line segment?

**(b)** Suppose we are given a regular $n$-gon (where $n$ is a positive integer greater than or equal to 4), and we choose four distinct vertices $A$, $B$, $C$, and $D$ of the $n$-gon at random. What is the probability that the vertices $A$ and $B$ lie on opposite sides of the line $\overleftrightarrow{CD}$?

**(c)** Consider the same unit square described in part (a). This time, suppose we choose numbers $k$ and $l$ at random from the interval $[0, 1]$, and draw the line segment from $(k, 0)$ to $(l, 1)$. We also choose two points $A$ and $B$ at random in the interior of the given unit square. What is the probability that $A$ and $B$ lie on opposite sides of the randomly-chosen line segment?

**(d)** Suppose we are given a unit circle, and we choose two points $A$ and $B$ at random in the interior of the circle, and we choose two points $C$ and $D$ at random on the circumference of the circle.

- **i.** Write a computer program, in the language of your choice (computational algebra programs such as Maple and Mathematica are acceptable), that uses random numbers to simulate this experiment and obtain an approximate value of the probability that the points $A$ and $B$ lie on opposite sides of the line $\overleftrightarrow{CD}$. Include a copy of your code in your solution, and write notes explaining how the program works.

- **ii.** Determine the exact value of this probability.

**Problem 5: Digisum Frontier**

Given a positive integer $n$, we define the *digisum* of $n$, written $d(n)$, to be the sum of the base-10 digits of $n$. (So for example, $d(42) = 4 + 2 = 6$, $d(300) = 3 + 0 + 0 = 3$, and $d(1337) = 1 + 3 + 3 + 7 = 14$.)

Given a set $S$ of positive integers, we define the *digisum set* $D(S)$ to be the set of all positive integers that are digisums of integers in $S$. We write this in set notation as $D(S) = \{d(n) : n \in S\}$. So for example, we have $D(\{19, 80\}) = \{8, 10\}$, and $D(\{26, 53, 71\}) = \{8\}$.

We say that a set $S$ is *digisum-ascending* if $S$ is a subset of $D(S)$ (possibly equal to $D(S)$), and *digisum-descending* if $D(S)$ is a subset of $S$ (possibly equal to $S$). A set $S$ is *digisum-stable* if $S = D(S)$.

In this problem, we will explore some interesting facts about digisum sets.

**(a)** Compute, with proof, each of the following:

- **i.** $D(S)$, where $S$ is the set of all nine-digit positive integers.
- **ii.** $D(S)$, where $S$ is the set of even positive integers.
- **iii.** $D(S)$, where $S$ is the set of all positive multiples of 2010.

**(b)** Give an example of a set that is neither digisum-stable nor digisum-ascending nor digisum-descending.

**(c)** Find, with proof, all finite sets of positive integers that are digisum-ascending.

**(d)** Find, with proof, all finite sets of positive integers that are digisum-stable.

**(e)** Given a set $S$ of positive integers, define $D_0(S) = S$, and for $n \geq 1$, define $D_n(S) = D(D_{n-1}(S))$. Is there a set $S$ of positive integers such that $D_{n-1}(S)$ is a proper subset of $D_n(S)$ for every positive integer $n$?

**(f)** Suppose that $S$ is a digisum-ascending set and $T$ is a digisum-descending set such that $S \subseteq T$. Prove that there is a digisum-stable set $U$ such that $S \subseteq U \subseteq T$.