Saturday Morning Math Group – Austin Math Circle
Austin Area Problem Solving Challenge

Rules

1. The Austin Area Problem Solving Challenge (AAPSC) is a competition for teams of **up to five students each**. Any student who resides in the Austin area and has not yet graduated from high school is eligible. Teams may include students from different schools.

2. This problem set consists of five problems. Each problem is worth a total of 40 points, but a solution that goes above and beyond what is asked in the problem may earn more than 40 points.

3. You may not discuss any of these problems, or any directly related problems, with people other than those on your team until after the problem set is due. However, you may use books, notes, and the Internet to look up mathematical ideas that you think may help with the problems (but not to ask for assistance with the problems themselves). If you need clarification regarding any of the problems, please e-mail Cody L. Patterson at smmg@math.utexas.edu.

4. While working on the problems, you may use technology (*e.g.* calculators, computers) to perform computations and experiments. However, you must explain how you arrived at each of your solutions; and solutions that can be reproduced and/or obtained without technology will earn higher marks than solutions that require technology.

5. You may submit your solutions in any of the following ways:

- Electronically, to smmg@math.utexas.edu (All electronic submissions must be in .pdf or .doc format)
- By mail, to Saturday Morning Math Group, Department of Math, UT Austin, 1 University Station C1200, Austin, TX 78712
- At the April 12 Math Circle meeting

All solutions must be received by 10 AM, April 12.

Guidelines for Solutions

1. Please fill out the **AAPSC Cover Page** and attach it to the front of your solutions packet. **Do not identify yourself or your teammates in any way in the solutions themselves.**

2. Please write all solutions in complete sentences, and be neat. Partial credit will be awarded for incomplete solutions if these solutions are well-written and contain potentially useful ideas.

3. Solutions may be handwritten or typed. If you wish to submit your solutions electronically, you must use .pdf or .doc format.

4. Please send all of your solutions in one submission; do not send several separate packets (or send part of your work by mail and the rest of it electronically).

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Cover Sheet

	Name	**E-mail address**
Team		
Team Member #1		
Team Member #2		
Team Member #3		
Team Member #4		
Team Member #5		

Notes:

1. Please come up with a team name that is unique; your team name may include the name of your school (if all team members are from the same school), but it should also contain something that distinguishes you from other teams from your school that might submit solutions.

2. Please include a working e-mail address for each team member. If your team wins a prize, we will need to contact you and ask you which of several available prizes you want.

3. If you send your solutions via e-mail, you do not need to include this Cover Sheet; instead, please provide all of the requested information in the body of your e-mail, and include the solutions as attachments.

Good luck, and enjoy the problems!

Problem 1: The Outward Spiral

Alice gets out an infinitely long, infinitely wide sheet of paper and begins writing the positive integers, in order, in a spiral as shown below:

17	16	15	14	13
18	5	4	3	12
19	6	1	2	11
20	7	8	9	10
21	22	23	24	25...

Alice continues writing numbers indefinitely, continuing outward from the center. For purposes of this problem, suppose that she has written all of the positive integers, forming an infinite array of numbers.

(a) If we begin at the number 1 and start moving directly to the right, the first numbers we come across are 1, 2, and 11. What are the next three?

(b) As we move directly to the right from the number 1 (as we did in part (a)), we encounter a sequence of numbers, beginning with the numbers 1, 2, and 11. If we define a_n to be the n^{th} number in this sequence, with $a_1 = 1$, $a_2 = 2$, $a_3 = 11$, and so on, give a formula for a_n in terms of n.

(c) If we move five numbers to the right of the number 1 and three numbers down, what number do we land on?

(d) If we move thirteen numbers up from the number 1 and nine numbers to the left, what number do we land on?

(e) In Alice's array, is the number 2008 immediately above the number 2007, immediately below the number 2007, immediately to the right of it, or immediately to the left of it? Prove your answer.

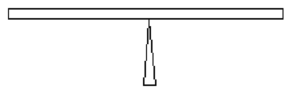
(f) Consider the diagonal through the number 1 that contains the numbers 17, 5, 1, 9, and 25. Suppose Alice computes the sum of 201 numbers on this diagonal: the 1 in the center, the first hundred numbers on the diagonal above and to the left of it, and the first hundred numbers on the diagonal below and to the right of it. What sum should she get?

(g) Consider the row of the array containing the numbers 19, 6, 1, 2, and 11. Prove that this row does not contain any numbers that are divisible by 21.

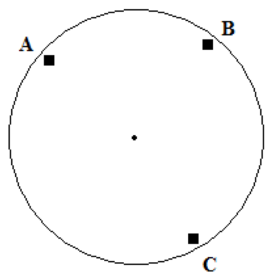
Problem 2: Weights on a Plate

(*Note*: Feel free to use calculators on this problem.)

A thin, weightless, circular plate balances on a needle placed exactly underneath the center of the plate (side view shown below):



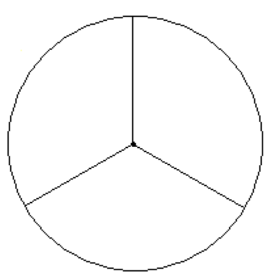
Flynn places three point masses (that is, weights with negligible volume) on the circumference of the plate. Weights A and B are separated by a $92°$ arc, weights B and C are separated by a $113°$ arc, and weights C and A are separated by a $155°$ arc. (See figure below.) It turns out that when the weights are placed precisely in this configuration, the plate balances perfectly.



(a) Rank the three weights A, B, and C from lightest to heaviest. Prove that your ranking is correct.

(b) Suppose that Flynn places weight A on the plate 10 centimeters directly to the north of the center and places weight B on the plate 7 centimeters directly to the south of the center. If the radius of the plate is 15 centimeters, is it possible for Flynn to place weight C on the plate so that the plate balances? If so, where should he put it?

(c) Now suppose that Flynn draws three rays on the plate. One ray points directly north from the center; the other two rays meet the first ray at the center at $120°$ angles, as shown below. Is it possible for Flynn to place the three weights on the rays (one on each ray) so that the plate balances? Prove your answer.



(d) Suppose that Flynn removes the weights from the plate, and now places three new weights X, Y, and Z on the circumference of the plate so that the plate balances perfectly. Weights X and Y are separated by an arc of p degrees, weights Y and Z are separated by an arc of q degrees, and weights Z and X are separated by an arc of r degrees. If we know that $0 < p < q < r < 180$ but do not know anything else about p, q, and r, can we determine (based on this knowledge alone) which weight is the heaviest and which is the lightest?

Problem 3: Digital Detective

In this problem, we use the word "string" to mean a sequence of digits that appear consecutively in a number, in the order given. For example, the number 1234567 contains the string "34567", but does not contain the string "245", since these digits do not appear consecutively in the number. In this problem, we'll be looking at what strings can be found in multiples of a given integer, and what strings can be found in perfect n^{th} powers for a given value of n.

(a) Give an example of each of the following:

1. A multiple of 3 that contains the string "1492".
2. A multiple of 7 that contains the string "1776".
3. A multiple of 23 that contains the string "2008".
4. A multiple of 867 that contains the string "5309".

(b) Suppose that "$d_1 d_2 \cdots d_k$" is a string of digits, and n is a positive integer. Prove that there exists a positive multiple of n that contains the string "$d_1 d_2 \cdots d_k$".

(c) Suppose that "$d_1 d_2 \cdots d_k$" is a string of digits, and n is a positive integer that is not divisible by 2 or 5. Prove that there exists a positive multiple of n that ends with the string "$d_1 d_2 \cdots d_k$" (on the right). (*Hint:* You may find the following theorem useful: If a and b are integers such that the greatest common divisor of a and b is 1, then there exist integers x and y such that $ax + by = 1$.)

(d) Explain why it is necessary to assume in part (c) that n is not divisible by 2 or 5.

(e) Consider the numbers

$$10001^2,\ 10002^2,\ 10003^2,\ 10004^2,\ \ldots,\ 10049^2.$$

Does at least one of these numbers contain the string "08"? How about the string "54"? How about "22"?

(f) Give an example of a positive perfect cube that contains the string "2008". Is there a positive perfect cube that ends with the string "2008" (on the right)?

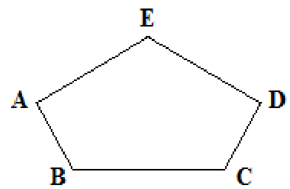
(g) Suppose that "$d_1 d_2 \cdots d_k$" is a string of digits, and n is a positive integer greater than 1. Prove that there exists a positive perfect n^{th} power (that is, an integer of the form a^n where a is a positive integer) that contains the string "$d_1 d_2 \cdots d_k$".

Problem 4: Pentagon Briefing

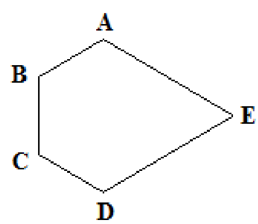
In this problem, to *tile* the plane is to cover the plane with polygons in such a way that, if two polygons touch each other, then they touch only at a vertex, or along a full edge. (In other words, it is not allowed for the interiors of two polygons to overlap, or for a full edge of one polygon to coincide with part of an edge of another polygon.) Unless otherwise specified, the polygons do not have to be congruent to one another; in fact, it is allowed for the polygons to have infinitely many different shapes.

(a) Explain why it is impossible to tile the plane with regular pentagons.

(b) Let $ABCDE$ be a pentagon such that $AB = 1$, $BC = 2$, $CD = 1$, $DE = \sqrt{3}$, and $EA = \sqrt{3}$; and $m\angle A = 90°$, $m\angle B = 120°$, $m\angle C = 120°$, $m\angle D = 90°$, and $m\angle E = 120°$. (See the figure below.) Is it possible to tile the plane with copies of the pentagon $ABCDE$? Prove your answer.



(c) Let $ABCDE$ be a pentagon such that $AB = BC = CD = 1$ and $DE = EA = 2$; and $m\angle A = m\angle B = m\angle C = m\angle D = 120°$ and $m\angle E = 60°$. Is it possible to tile the plane with copies of the pentagon $ABCDE$? Prove your answer.



(d) Prove or disprove that it is possible to tile the plane with convex pentagons so that every interior angle of every pentagon is either a right angle or a 135° angle.

(e) Prove or disprove that it is possible to tile the plane with convex pentagons so that every interior angle of every pentagon is obtuse.

Problem 5: Is the Shortest Way the Fastest Way?

Perry the particle lives in the coordinate plane; he always stays in the first quadrant. When he wants to move from one place to another, he is allowed to move in the following ways:

- Directly to the left or right, at a speed (in units per second) equal to Perry's current y-coordinate.
- Directly up or down, at a speed (in units per second) equal to Perry's current x-coordinate.

So for example, if Perry moves from the point $(4, 6)$ directly to the point $(7, 6)$, then he moves a distance of 3 units at a speed of 6 units per second; therefore, this movement takes one half of a second. If he moves from the point $(1, 10)$ directly to the point $(1, 3)$, then he moves a distance of 7 units at a speed of 1 unit per second; therefore, this movement takes 7 seconds.

Note: Perry is not required to move an integral number of units on a given move; he may, for example, move from the point $(3, 4)$ to $(3, 5.3)$.

(a) Suppose Perry starts at the point $(2, 2)$ and moves along the following sequence of points: $(3, 2)$, $(3, 5)$, $(1, 5)$, $(1, 7)$, $(6, 7)$, $(6, 1)$. Assuming that he goes directly from each point in this sequence to the next, how long does it take for Perry to complete this sequence of moves?

(b) Suppose that x_1, x_2, y_1, and y_2 are positive real numbers such that $y_1 < y_2 < x_1 < x_2$. If Perry wants to move from the point (x_1, y_1) to the point (x_2, y_2), which of the following is faster: for Perry to move from (x_1, y_1) directly to (x_2, y_1) and then directly to (x_2, y_2); or for him to move from (x_1, y_1) directly to (x_1, y_2) and then directly to (x_2, y_2)?

For parts (c) through (e), Perry wants to travel from the point $(1, 1)$ to $(2, 2)$ as quickly as possible. He may do this in any finite number of steps, but all steps must be rightward or upward. Note that this means that Perry's path will consist of alternating rightward moves and upward moves; after each rightward move, Perry turns left (in order to move upward); and after each upward move, Perry turns right.

(c) Suppose that Perry takes a particular path P from $(1, 1)$ to $(2, 2)$. Show that there exists a path P' that Perry can take that satisfies the following properties:

1. The path P' does not take any more time to travel than P.
2. The path P' does not cross (but is allowed to touch) the line $x = y$.

(d) Suppose that Perry takes a particular path P from $(1, 1)$ to $(2, 2)$ such that P does not cross the line $x = y$. Prove that there exists a path P' that Perry can take that satisfies the following properties:

1. The path P' does not take any more time to travel than P.
2. Each time the path P' turns to the right, it does so on the line $x = y$.

(e) How quickly can Perry get from the point $(1, 1)$ to the point $(2, 2)$? Prove your answer.

(f) Suppose Perry wants to travel from the point $(1, 4)$ to $(1, 8)$. Again, he may do this in any finite number of steps; but this time, the steps may be in any direction. How quickly can Perry reach his destination?