COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 10

Due Friday, April 19, 2013, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Consider the hyperbolic plane, that is, the upper half plane $\mathbb{H}$ endowed with the Poincaré metric, $ds^2 = \frac{d\bar{z}dz}{(Imz)^2}$. The distance between any pair of points $z_1, z_2 \in \mathbb{H}$ is obtained from minimizing the arc length,

$$d(z_1, z_2) = \inf_{\gamma} \int_0^1 \frac{|\dot{\gamma}(t)|}{Im(\gamma(t))} dt \,,$$

where the infimum is taken over all C^1-curves $\gamma : [0,1] \to \mathbb{H}$ connecting z_1, z_2 with $\gamma(0) = z_1$, $\gamma(1) = z_2$. Curves of minimal arc lengths are called geodesics.

(i) Prove that automorphisms of the hyperbolic plane are isometries. That is,

$$d(T_A(z_1), T_A(z_2)) = d(z_1, z_2)$$

for all Möbius transformations $T_A(z) = \frac{az+b}{cz+d}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{R})$.
Hint: Prove that the Poincaré metric is invariant under automorphisms of $\mathbb{H}$.

(ii) Prove that straight lines in $\mathbb{H}$ perpendicular to the real line are geodesics.

(iii) Prove that half circles in $\mathbb{H}$ with centers on the real line are geodesics. Moreover, prove that there are no other geodesics apart from those in (ii).

Hint: For the first part in (iii), consider the action of automorphisms of $\mathbb{H}$ on the geodesics found in part (ii). For the second part in (iii), use the stereographic projection as an auxiliary tool.

2. Problem

Let $L := \{m + in \,|\, m, n \in \mathbb{Z}\}$, and $L^* := L \setminus \{(0,0)\}$.

(i) Prove that

$$f(z) := \frac{1}{z^2} + \sum_{\omega \in L^*} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big)$$

defines a meromorphic function in $\mathbb{C}$ satisfying $f(z + \omega) = f(z)$ for all $\omega \in L$.

(ii) Verify that f defines an analytic map $\mathbb{T} \to \mathbb{C}_\infty$, from the torus $\mathbb{T}$ to the Riemann sphere. Determine the degree of f, and find branch points and their valencies, if there are any. Compare your results with the Riemann-Hurwitz formula.