COMPLEX ANALYSIS – PRACTICE PROBLEMS

1. Problem

(a) Prove that if $f \in \mathcal{H}(\Omega)$, then $g(z) := \overline{f(\frac{1}{\bar{z}})}$ is holomorphic in $\overline{\Omega} = \{z \in \mathbb{C} \,|\, \bar{z} \in \Omega\}$. In particular,

$$g'(z) = \overline{f'(\bar{z})}\,.$$

(b) What is the general form of a rational function $R = \frac{P}{Q}$ (where P, Q are polynomials) which has absolute value 1 on the unit circle $|z| = 1$? In particular, how are the zeros and poles related to each other ?
Hint: Consider $h(z) := R(z)\overline{R(\frac{1}{\bar{z}})}$.

2. Problem

Assume that $f \in \mathcal{H}(\Omega)$, and let γ be a closed contour in Ω. Prove that $\oint_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary.

3. Problem

Assume $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial of degree $n > 0$. Assume that $|P(z)| \leq 1$ for $|z| = 1$. Prove that then, $P(z) = z^n$.