1. Find all the complex numbers described by the following formulas and sketch their location in the complex plane.

   a \((-i)^{2/3}\).

   b \(\left(\frac{1+i}{1-i}\right)^{1/2}\).

   c \(1^i\).

   d \(\log(-i/3)\).
2. Compute the following limits

a \( \lim_{n \to \infty} \frac{n^2}{n^2 + 2n + 3} \)

b \( \lim_{z \to -3i} \frac{z + 3i}{z^2 + 9} \)

c \( \lim_{z \to 2} \frac{e^z}{z^2 - 4} \)

d \( \lim_{z \to 0} \frac{1 - \cos z}{z^2} \).
3. Find the integral of the function $f(x, y) = 3x^2 + ixy$ along the contour formed by the straight line from $(-2, 1) = -2 + i$ to $(2, -1) = 2 - i$.

4. Let $C$ be a simple closed contour in the plane. Use Green’s theorem to show that $\int_C \bar{z} \, dz = 2i \text{Area}(Q)$. Here $Q$ is the region bounded by $C$. Illustrate the theorem by letting $C$ be the circle of radius $R$ about the origin and computing both the contour integral and the area separately.
5 Compute the following integrals: explain in each case whether you use the properties of anti-derivatives, Cauchy’s theorem, or one of the Cauchy integral formulas.

a $\int_C \frac{1}{z} \, dz$ where $C$ is the unit semi-circle from $-i$ to $i$.

b $\int_C 2^z \, dz$ where $C$ is the circle of radius 2 about 2 and $\arg 2 = 0$.

c $\int_C \frac{dz}{z^2 - 9}$ where $C$ is the same circle as in part b).

d $\int_C \frac{\cos z}{(z - \pi)^3} \, dz$ where $C$ is again the circle as in part b).
6.

T F Every function \( f(x, y) = u(x, y) + iv(x, y) \), where \( u \) and \( v \) are defined and smooth in the plane, is analytic.

T F There is a function \( f(z) \) which is analytic in, for which \( |f(e^{i\theta})| \leq 1 \) and \( f(0) = i + 1 \).

T F If \( f(z) = u(x, y) + iv(x, y) \) is analytic, then \( v(x, y) \) is harmonic.

T F If \( f(z) = u(x, y) + iv(x, y) \) and \( u \) and \( v \) are harmonic, then \( f \) is analytic.
7. Prove the fundamental theorem of algebra: Every polynomial $P(z)$ of degree $n > 0$ can be factored as the product of $n$ linear factors $z - z_i$ and a constant.

a Every polynomial $P(z)$ of degree $n > 0$ has at least one point $z_0$ where $P(z_0) = 0$. Assume this is false and find a contradiction by looking at the properties of $\frac{1}{P(z)}$.

b Prove that if $P(z_0) = 0$, then $\frac{P(z)}{z - z_0}$ has a removeable singularity at $z_0$. 
c Now show that \( \frac{P(z)}{z - z_0} \) is a polynomial of degree \( n - 1 \). Either follow the suggestion in the book, or use the Cauchy integral formulas and the behavior of this function for large \( |z| \).

d Finish the proof by using induction on the degree of the polynomial.
8. a  Expand \( \log \frac{1+z}{1-z} \) in a Taylor series around \( z = 0 \). Use the principle value of log for your computations.

b  Expand \( \frac{z+3}{z^2+z} \) in a Laurent series about \( z = -1 \).
9. Evaluate \( \int_{0}^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} \) for \( a > 1 \).

10. Evaluate \( \int_{-\infty}^{\infty} \frac{e^{iax}dx}{(1 + x^2)(4 + x^2)} \) for \( a \geq 0 \).
Extra Credit

11. Express the Cauchy Riemann equation in coordinates $t, s$, where $x = e^t \cosh s, y = e^t \sinh s$. Use an extra page if necessary.

12. Evaluate $\int_0^\infty \frac{x^a}{1+x^2} \, dx$, where $-1 < a < 1$. 