ALGEBRA FINAL

F'06

Due 12/12/06
9:00 am

Name: ____________________

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1. Let $H \leq G$ be a subgroup of finite index n. (i) Show that there exists a normal subgroup $N \trianglelefteq G$ of finite index with $N \leq H$. (ii) Give explicitely (in terms of n) a number $m>0$ such that

$$g^m \in H \text{ for all } g \in G.$$

2. Let G be a finite group, p a prime with $p \mid |G|$.

   i) Show that $p \mid \#\{g \in G \mid g^p = 1\}$

   ii) Show that
   $$\#\{H \leq G \mid |H| = p\} \equiv 1 \bmod p$$

   (iii) **BONUS** Prove that for every $1 \leq k$ such that $p^k \mid |G|$ we have
   $$\#\{H \leq G \mid |H| = p^k\} \equiv 1 \bmod p$$

HONOR CODE
I pledge my honor that this is my own work done with no consultation of books, notes, papers, websites, people other than the class notes and Milne / Ash.

Signature: ____________________ Date: ____________

3. i) Let G be a finite group, n_p the number of p Sylow subgroups of G for some prime $p \mid |G|$. Prove that if $n_p \not\equiv 1 \bmod p^2$ then there are distinct p-Sylow subgroups P_1, P_2 of G ~~a~~ such that $[P_1, P_1 \cap P_2] = [P_2, P_1 \cap P_2] = p$

ii) Show there are no simple groups of order $3159 = 3^4 \cdot 13$

4. Let R be a commutative ring with 1 such that all submodules of a free module are free. Prove that R is a PID (I.e. show that R is a domain and every ideal of R is principal).

5. Decide whether the following two matrices are similar (over $\mathbb{Q}$)

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}$$

6. Let R be a PID. Let N_1, N_2, M be R-modules such that

$$N_1 \oplus M \simeq N_2 \oplus M$$

prove that $N_1 \simeq N_2$.