HANDOUT 4

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1. Merlin's Magic Square

Merlin's magic square is one of several commercial puzzles based on the same principle (others include *Lights out, Fiver*, etc.) The puzzle consists on a number of cells or buttons that can be on or off, or in general could be of one of a fixed number of colors. Clicking a given cell will change the colors of some of the cells according to some pre-established rule (figuring out the rule could be part of the puzzle), which only depends on that cell and not on the particular status of the puzzle. For example, one version could be, say, six buttons arranged in a circle in which clicking any button changes cyclically the colors of its two immediate neighbors. Typically, the goal of the puzzle is to turn all cell off starting from some prescribed configuration (for example all cells on in the on/off case) or some random one.

More generally, we can play this puzzle on any directed graph Γ by the rule that clicking on a vertex (representing a cell) changes the colors of its successors cyclically. (The successors of a vertex w are those vertices u with $w \mapsto u$.)

Let m be the number of colors. Label these colors cyclically $0, 1, \ldots, m-1$ with the label read modulo m so that colors m and 0 are the same, the colors $(m+1)$ and 1 are the same, etc. With this notation to *change colors cyclically* means simply to *add* 1 to the color. We let $\mathbb{Z}/m\mathbb{Z}$ denote the integers modulo m.

A *configuration* or *state* of the puzzle is a description of the status of all cells at a given time. We can describe a state as a function

$$s : \quad \Gamma \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

which at a vertex $v \in \Gamma$ has the value $s(v) \in \mathbb{Z}/m\mathbb{Z}$ giving the current color of the cell v.

The effect of clicking on a vertex $w \in \Gamma$ is to add 1 (i.e. change color cyclically) to the value of s at the successors of w. Algebraically, we can describe the new state as

$$s + c_w \bmod m$$

where

$$c_w(u) = \begin{cases} 1 & w \mapsto u \\ 0 & \text{otherwise} \end{cases}$$

If we let s_I and s_F be the intial and final states respectively, to solve the puzzle means to find a sequence of vertices $w_1, \ldots, w_k$ such that

$$s_F \equiv s_I + c_{w_1} + \cdots + c_{w_k} \bmod m \tag{1}$$

We can draw a number of interesting conclusions from this analysis. First, since addition modulo m is commutative (we have already implicitly used that it is associative to write (1) without any brackets) we see that *(i) It does no matter in what order we click a given sequence of buttons, the final result will be the same.*

Moreover, since adding any number m times will give zero modulo m we also have that

(ii) Clicking a given button a total of m times is, independently of all other clicks in between, the same as not having clicked on it at all.

We can write our basic equation (1) in a slightly different way. Let us order the vertices of Γ as $v_1, \ldots, v_n$ where n is the total number of vertices. We can represent the state function simply as a vector $(s_1, s_2, \cdots, s_n)$ where $s_i \in \mathbb{Z}/m\mathbb{Z}$ is the color of v_i.

The c_v's are themselves vectors of the same kind and another way to write (1) is

$$s_F \equiv s_I + At \bmod m \tag{2}$$

where A is the matrix whose j-th column is the vector c_{v_j} describing the successors of the j-th vertex v_j and the i-th entry of $t = (t_1, t_2, \ldots, t_n)$ is the number of times we click on the i-th button.

Notice that A is the incidence matrix of Γ; that is, the (i, j) entry of A is 1 if $v_j \mapsto v_i$ and is 0 otherwise. To give A is basically the same as giving Γ.

We can now answer the following question: will it be possible to solve the puzzle for arbitrary choices of s_I and s_F? In light of (2) this question is equivalent to the following. Given any vector s can we always find t such that

$$s \equiv At \bmod m? \tag{3}$$

The answer to this question is given by the theory of arithmetic of integers modulo m: We may always solve (3) if and only if $A \bmod m$ is invertible, which in turns is equivalent to the condition that $\det(A)$ be invertible modulo m. Think of A as matrix with plain integers (as opposed to integers modulo m) and let $d = \det(A)$, which is now an integer. Then, we have our final criterion

(iii) The puzzle can be solved for any intial and final states if an only if $d = \det(A)$ is relatively prime to m.

In fact, if this condition holds we may compute an inverse A^{-1} of A modulo m and we can then solve (2) by

$$t = A^{-1}(s_F - s_I), \tag{4}$$

in particular the solution is unique and we also find that

(iv) If d is relatively prime to m then the sequence of moves that solves the puzzle is unique (modulo m).

Moreover, if (iii) fails there will always be final states that cannot be reached and there will also be non-trivial sequences of moves that will take you back to the original state (so when a solution exists it is not unique).

2. An example

To illustrate the ideas of the previous section let us consider the example mentioned there of a puzzle consisting of six cells arranged in a circle so that clicking a button changes cyclically the color of its two immediate neighbors.

The adjacency matrix for the corresponding graph is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

of determinant $d = -4$. According to our criterion (iii) of the previous section the puzzle will be always solvable if and only if the number of colors m is odd.

The inverse of A as a matrix with rational coefficients is

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

Say we have $m = 3$ colors W white, R red and B blue. Since $2 \equiv -1 \bmod 3$ the inverse of A modulo 3 is then

$$A^{-1} \equiv - \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \bmod 3.$$

Suppose the cells in the puzzle have colors W, R, B, W, R, B and our goal is to turn them all white. We may label the colors $W = 0, R = 1, B = 2$ and then

$$s_I = (0, 1, 2, 0, 1, 2), \qquad s_F = (0, 0, 0, 0, 0, 0).$$

Using (4)

$$t \equiv A^{-1}(s_F - s_I) \bmod 3.$$

Doing this calculation we find

$$t = (0, 2, 1, 0, 2, 1).$$

This means that (modulo 3) the only way to solve this case of the puzzle is to click twice on buttons 2 and 5 and once in buttons 3 and 6.

If the number of colors is $m = 2$ on the other hand, the matrix A is *not* invertible and we cannot use (4). For some initial states we will be able to solve the puzzle but not for all.

To analize what happens in this case we will use basic results of linear algebra. Since the integers modulo 2 form a *field*, i.e., every non-zero element has a multiplicative inverse, most theorems of standard linear algebra apply. Note that we are talking about a field with 2 elements!

The sum of the entries in every column of A is 2 hence $At \equiv 0 \bmod 2$ for $t = (1, 1, \ldots, 1)$. Therefore, clicking all six buttons once has no effect on the puzzle. In fact, there are exactly three such moves: $t = (1, 0, 1, 0, 1), (0, 1, 0, 1, 0, 1)$ and $t = (1, 1, 1, 1, 1, 1)$. This is because (as one can verify) the rank of A mod 2 is 4 and $(1, 0, 1, 0, 1), (0, 1, 0, 1, 0, 1)$ span its kernel (note that $(1, 0, 1, 0, 1) + (0, 1, 0, 1, 0, 1) = (1, 1, 1, 1, 1, 1)$).

Given a final state s_F for what initial states s_I can we solve the puzzle? The answer to this question is contained in (2): $s_F - s_I$ must be in the column space of A. Since the rank of A mod 2 is 4 this space has dimension 4.

A vector space of dimension n over our field has 2^n elements; hence, for any final state s_F there are exactly $2^6 - 2^4 = 48$ states for which the puzzle cannot be solved. Hence, for a random choice of s_I we have a $2^4/2^6 = 1/4$ chance of being able to solve the puzzle.

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