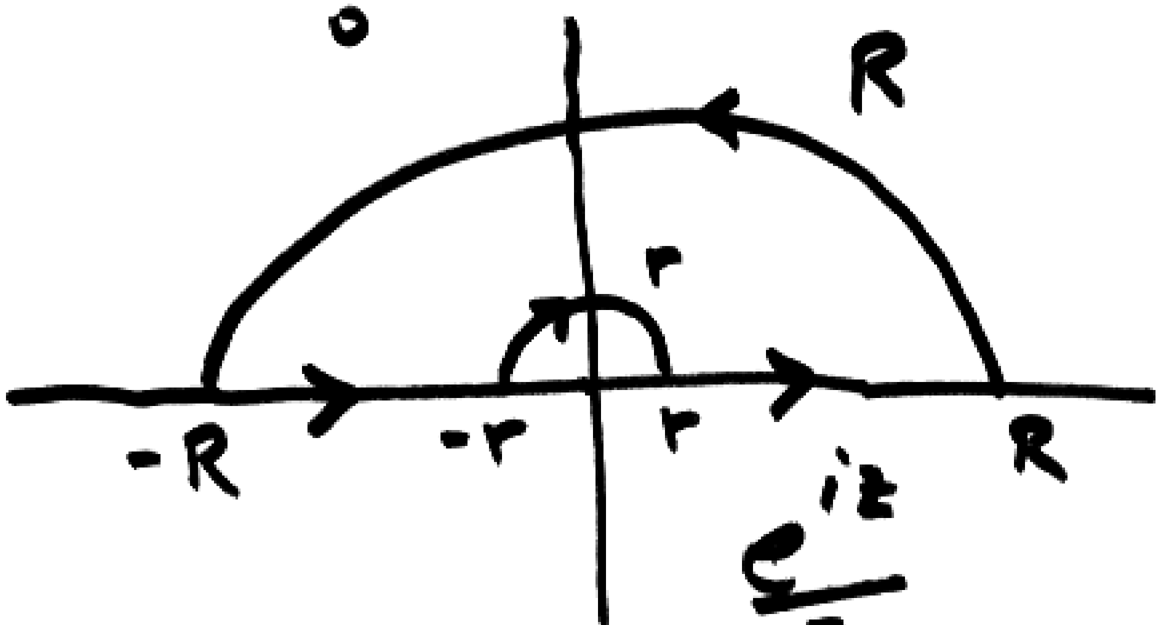
March 31, 2006

$$\int_0^\infty \frac{\sin x}{x}\,dx$$



$$0 = \int_\gamma \frac{e^{iz}}{z}\,dz$$

$$= \int_{C_R} + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_r^R \frac{e^{ix}}{x}\,dx + \int_{C_r}$$

$$f(z) = \frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z}$$

$$\int_{C_R} f(z)\, dz \longrightarrow 0$$

$R \to \infty$

by Jordan's lemma

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\, dx = -\int_{r}^{R} \frac{e^{-ix}}{x}\, dx$$

$$0 = \int_{C_R} + \int_{C_r} + \int_{r}^{R} \frac{e^{ix} - e^{-ix}}{x}\, dx$$

$$\underset{\downarrow}{\int_{C_R}}$$

$$0$$

$$\int_{C_r} \frac{e^{iz}}{z}\, dz$$

$$e^{iz} = 1 + z\, h(z)$$

h analytic at 0

$$\frac{e^{iz}}{z} = \frac{1}{z} + h(z)$$

$$\int_{C_r} \frac{e^{iz}}{z}\,dz = \int_{C_r} \frac{dz}{z} + \underset{\downarrow\, 0}{\int_{C_r} h(z)\,dz}$$

$$z = r e^{i\theta} \qquad 0 \le \theta \le \pi$$

$$\int_{C_r} \frac{dz}{z} = -i\int_0^{\pi} d\theta = -\pi i$$

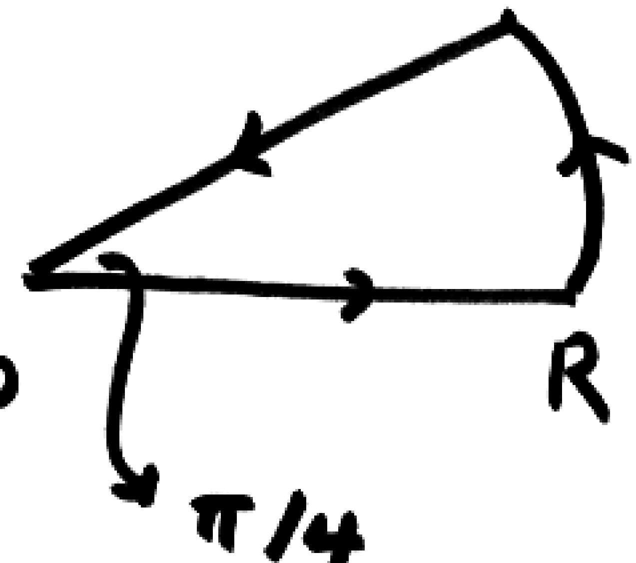
$$\underset{\downarrow}{2i\int_r^R \frac{\sin x}{x}\,dx} = -\underset{\substack{\downarrow \\ \pi i}}{\int_{C_r}} - \underset{\substack{\downarrow \\ 0}}{\int_{C_R}}$$

$$\int_r^R \frac{\sin x}{x}\,dx \rightarrow \boxed{\frac{\pi}{2}}$$

Fresnel integrals

$$\int_0^\infty \cos x^2 \, dx$$

$$f(z) = e^{i z^2}$$



$$\int_\gamma f(z)\, dz = 0$$

$$z = t\, e^{\frac{\pi i}{4}} \qquad dz = e^{\frac{\pi i}{4}} dt$$

$$i z^2 = i t^2 i = -t^2$$

$$f(z) = e^{-t^2}$$

$$0 = \int_0^R e^{i x^2} dx + \int_{C_R} e^{i z^2} dz \quad - \int_0^R e^{-t^2} dt \; e^{\pi i/4}$$

$$\left|\int_{C_R} e^{iz^2}dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta$$

$$z = R e^{i\theta}$$

$$z^2 = R^2 e^{i2\theta}$$

$$|e^{iz^2}| = e^{-R^2 \sin 2\theta}$$

$$0 \le \theta \le \pi/4, \quad 0 \le 2\theta \le \pi/2$$

$$\sin(2\theta) \geq \frac{2}{\pi} 2\theta$$

$$R\int_0^{\pi/4} e^{-R^2\frac{4}{\pi}\theta}\,d\theta$$

$$= R \cdot \frac{\pi}{R^2 4} \cdot e^{-R^2\frac{4}{\pi}\theta}\Big|_0^{\pi/4}$$

$$(1 - e^{-R^2})$$

$$\to 0 \quad \text{as } R \to \infty$$

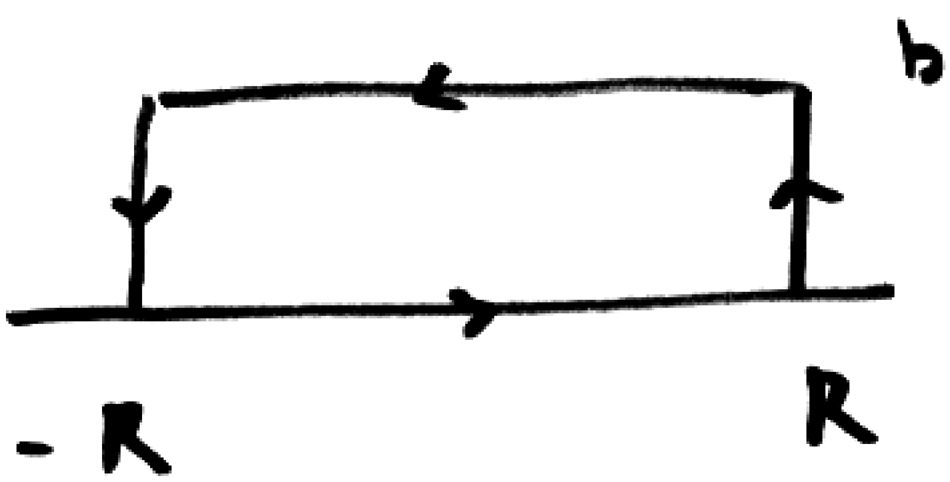
$$\int_0^\infty e^{ix^2}\,dx = e^{\pi i/4}\underbrace{\int_0^\infty e^{-t^2}\,dt}_{=\frac{\sqrt{\pi}}{2}}$$

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx =$$

$$e^{\pi i/4} = \frac{1+i}{\sqrt{2}}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2\pi}}{4}} =$$

$$\int_{-\infty}^{\infty} e^{-x^2} e^{2\pi a i x}\,dx$$



$$e^{-z^2}$$

$$z = x + ib$$

$$z^2 = x^2 + 2ibx - b^2$$

$$e^{-z^2} = e^{-x^2} e^{-2ibx} e^{b^2}$$

on top

$$-\int_{-R}^{R} e^{-x^2} e^{-2ibx} e^{b^2} \, dx$$

on bottom

$$\int_{-R}^{R} e^{-x^2} \, dx$$

vertical segments

$$z = R + it \qquad 0 \le t \le b$$

$$e^{-z^2} = e^{-R^2} \cdot e^{2Rit} \cdot e^{-t^2}$$

$$\int_{\substack{\text{vert} \\ \text{segment}}} \to 0$$