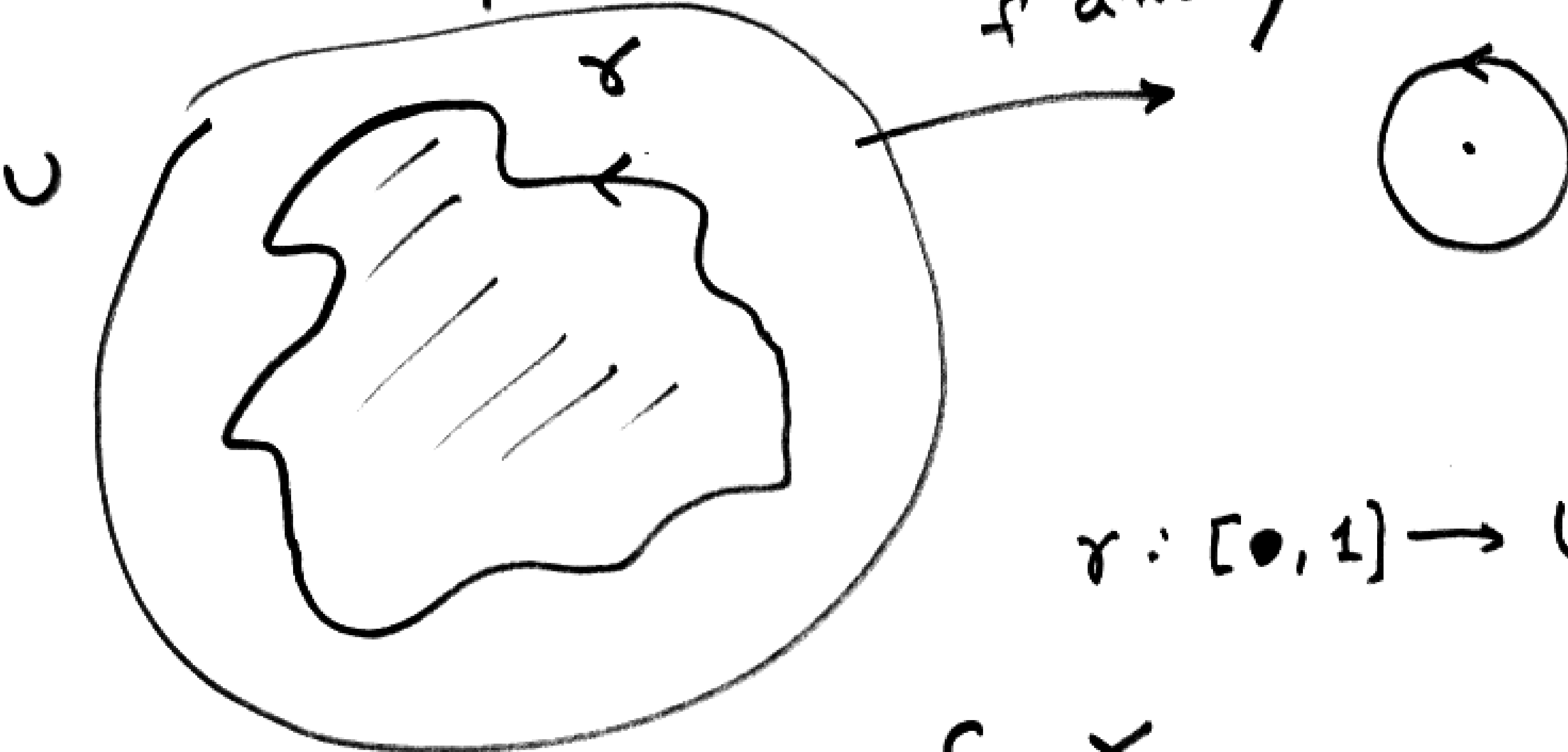
April 24, 2006

$p \in \mathbb{C}[z]$ deg n

$$L = \{ z \mid |p(z)| = 1 \}$$

$\mathbb{C} \setminus L$ has at most $n+1$ components.

f analytic non-constant



$\gamma : [0,1] \to U$

$$\sigma = f \circ \gamma$$

$|\sigma(t)| = 1$ all t

$\Rightarrow$ f has a zero inside γ

By argument principle equiv to

$$\# Z = n(\sigma, 0) > 0$$

If $\sigma'(t) \neq 0$ then we are done

$$\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

f' could vanish on γ !

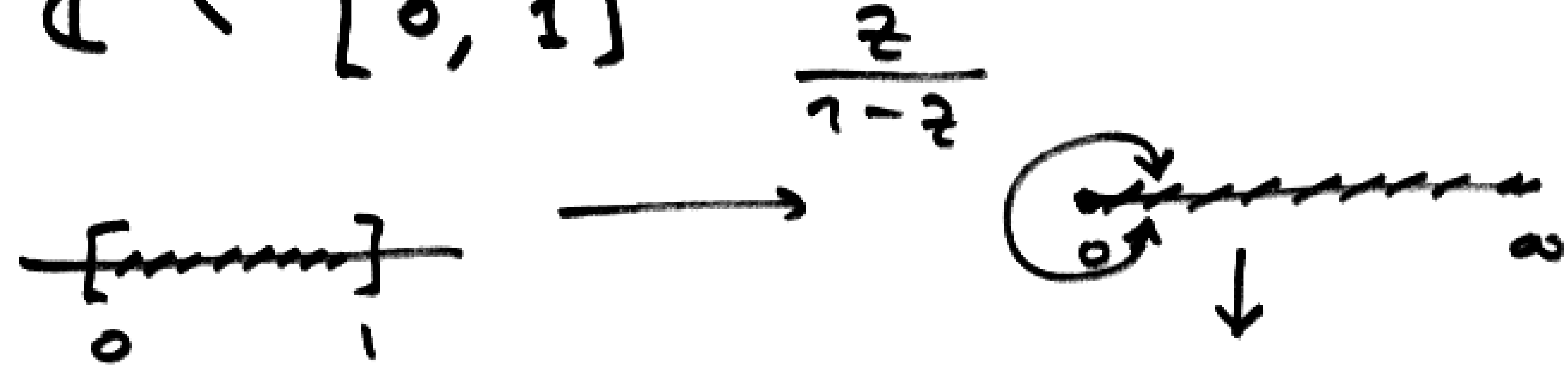
If f' does vanish on γ it does so (at finitely many points and we can deform γ) so that f' does not vanish.

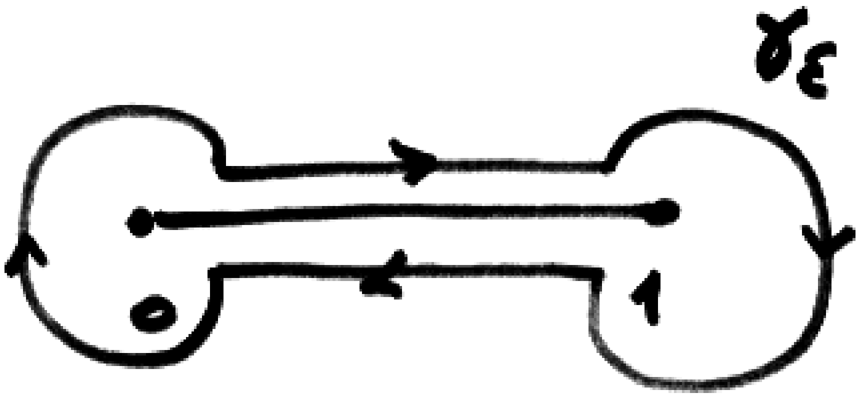
winding number of σ won't change.

to be completed...

$$f(z) = \left(\frac{z}{1-z}\right)^{\alpha} \frac{1}{z-a}$$

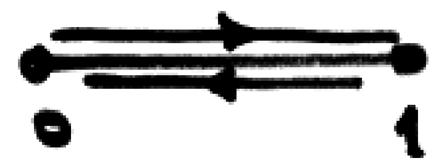
$\mathbb{C} \setminus [0, 1]$





$$\int_{\rightleftarrows} f(z)\,dz = \left(1 - e^{2\pi i \alpha}\right) I$$

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^{\alpha} \frac{dx}{x-a}$$

Shrink path $\xrightarrow[\varepsilon \to 0]{}$ 

$$\int_{\gamma_\varepsilon} f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=a} f + \operatorname*{Res}_{z=\infty} f \right)$$

$$w = \frac{1}{z}$$

$$\operatorname*{Res}_{z=a} f = \left(\frac{a}{1-a} \right)^{\alpha}$$

$$f(z) = \left(\frac{z}{1-z} \right)^{\alpha} \frac{1}{z-a}$$

$z = 1/w$

$$g(w) = f\left(\frac{1}{w}\right) = \left(\frac{1/w}{1-1/w} \right)^{\alpha} \frac{1}{1/w - a}$$

$$= \left(\frac{1}{w-1} \right)^{\alpha} \frac{w}{1-aw}$$

$$\cancel{\operatorname*{Res}_{z=\infty} f = \operatorname*{Res}_{w=0} g}$$

$$\frac{1}{2\pi i} \int_{|z|=R} f(z)\, dz = \operatorname*{Res}_{z=\infty} f$$

large R

$$= -\frac{1}{2\pi i} \int_{|z|=1/R} g(w) \frac{dw}{w^2}$$

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$$2\pi i\Big[\underset{\substack{\uparrow\\ \operatorname{Res}_{z=a} f}}{\left(\frac{a}{1-a}\right)^{\alpha}} - \underset{\substack{\uparrow\\ \operatorname{Res}_{z=\infty} f}}{e^{\pi i\alpha}}\Big] = (1-e^{2\pi i\alpha})\, I$$

$\alpha \neq 0$ $$I = \frac{2\pi i}{1-e^{2\pi i\alpha}} \cdot \left[e^{\pi i\alpha}\left(\frac{a}{a-1}\right)^{\alpha} - e^{\pi i\alpha}\right]$$

$$\left(\frac{a}{1-a}\right)^{\alpha} = \left(\frac{a}{a-1}\right)^{\alpha} e^{\pi i\alpha}$$

$a > 1$ negative real

$$\log \frac{a}{1-a} = \log\left|\frac{a}{1-a}\right| + i \underset{\substack{\parallel\\ \pi}}{\arg\left(\frac{a}{1-a}\right)}$$

$$\exp(\alpha\cdot\ldots) = \left|\frac{a}{1-a}\right|^{\alpha} e^{\pi i\alpha}$$

$$= \left(\frac{a}{a-1}\right)^{\alpha} e^{\pi i\alpha}$$

$\underline{z=0}$

$$\varepsilon e^{i\theta}$$

$$|f(\varepsilon e^{i\theta})| \leq \varepsilon^{\alpha} \cdot C$$

$$\left| \int_{\odot} f(z)\, dz \right| \leq C\, \varepsilon^{\alpha} \cdot \varepsilon$$

$\alpha + 1 > 0$

$\boxed{\alpha > -1}$

$$\xrightarrow[\varepsilon \to 0]{} 0$$

$$\int_0^{*} x^{\alpha}\, dx < \infty \quad \text{if} \quad \alpha > -1$$

$$\int_{\varepsilon}^{*} x^{\alpha}\, dx = -\frac{1}{\alpha+1}\varepsilon^{\alpha+1} + *$$

$\alpha + 1 > 0$ $\to 0$

same argument around 1

use $\boxed{\alpha < 1}$

$$\frac{g(w)}{w^2} = \left(\frac{1}{w-1}\right)^{\alpha} \cdot \frac{1}{w(1-aw)}$$

$$-\operatorname*{Res}_{w=0} \frac{g(w)}{w^2} = -\left(\frac{1}{-1}\right)^{\alpha} \cdot \frac{1}{1.}$$

$$= -e^{\pi i \alpha}$$

For our branch

$$\log(-1) = i \arg(-1)$$
$$= i\pi$$

$$\int \left(\frac{z}{1-z}\right)^{\alpha} \frac{1}{z-a}\, dz \quad \to 0$$
$$\varepsilon \to 0$$

$$\to 0$$

because $|\alpha| < 1$

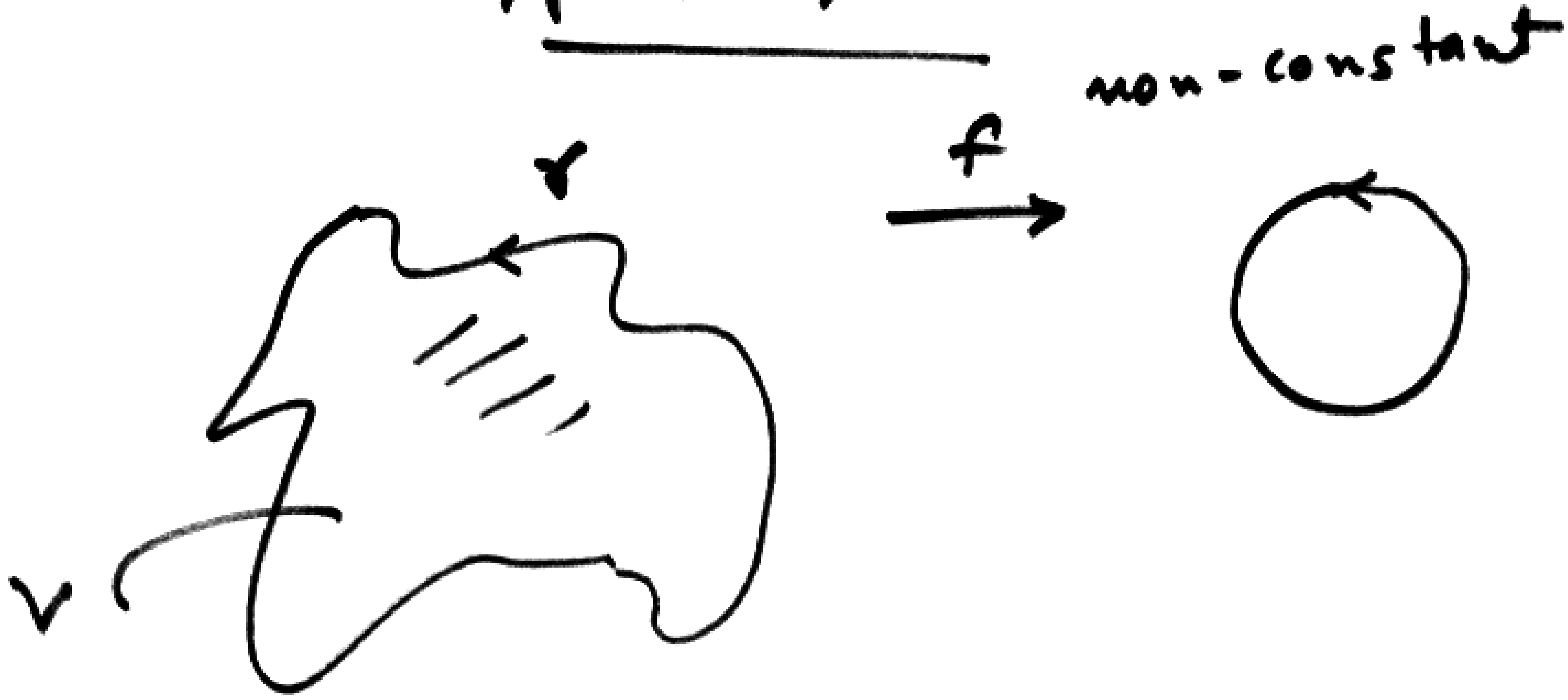
$$= \frac{2\pi i \, e^{\pi i \alpha}}{1 - e^{2\pi i \alpha}} \cdot \left(\left(\frac{a}{a-1} \right)^{\alpha} - 1 \right)$$

$$=$$

$$2\pi i \, \frac{e^{\pi i \alpha}}{1 - e^{2\pi i \alpha}} = \frac{2\pi i}{e^{-\pi i \alpha} - e^{\pi i \alpha}} = \frac{-\pi}{\sin \pi\alpha}$$

$$\boxed{I = \frac{\pi}{\sin \pi\alpha} \left(1 - \left(\frac{a}{a-1} \right)^{\alpha} \right)}$$

April 26, 2006



$$|f(z)| = 1 \qquad z \in \gamma$$

$\Rightarrow$ f vanishes inside $\gamma =: V$

max modulus principle

$$|f(z)| \le 1 \quad \text{on } V$$

$f \neq 0$ on V

$$g(z) = \frac{1}{f(z)} \quad \text{analytic}$$

on some open set containing $\overline{V}$

$$|g(z)| \le 1 \qquad z \text{ in } V$$

$\Rightarrow |f(z)| = 1$ in V

$\Rightarrow$ f is constant.

$$|g(z)| = 1 \qquad z \in \gamma$$

Pblm $\quad p \quad \deg n \geq 1$

$$L = \{ z \in \mathbb{C} \mid |p(z)| = 1 \}$$

$\mathbb{C} \setminus L$ has at most $n+1$ components.

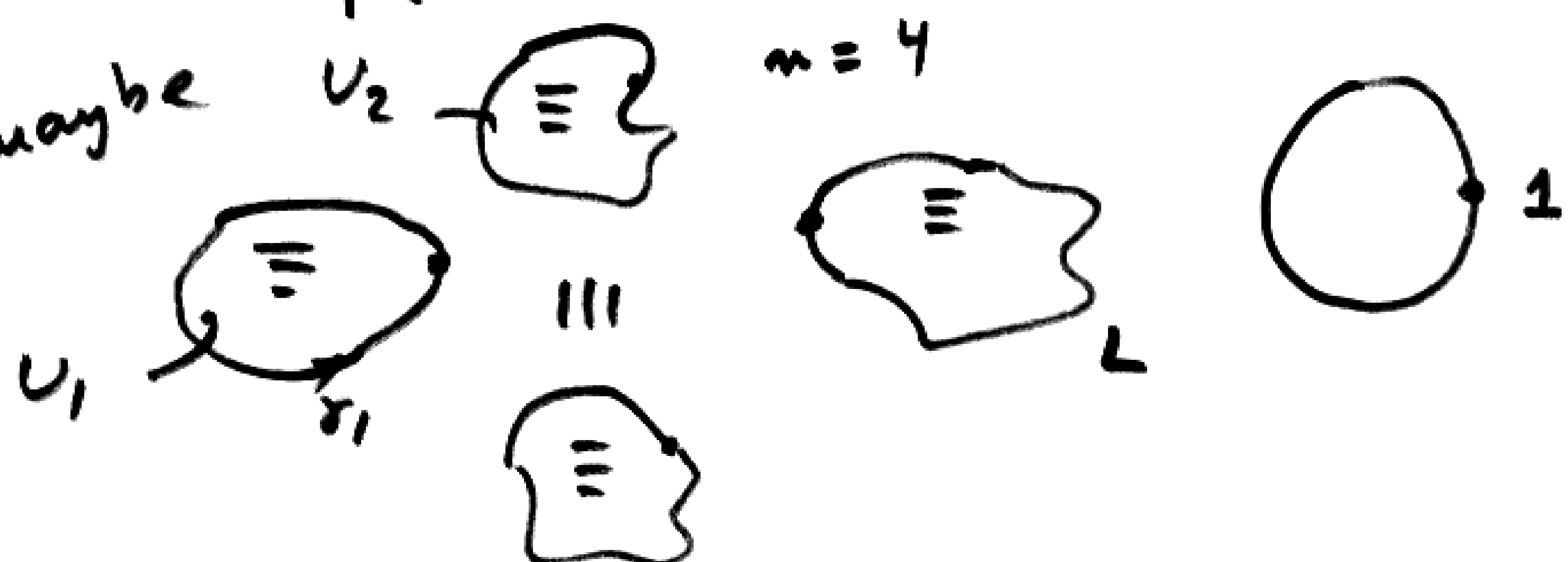
- $p(z) = z^n \qquad n \geq 1$

$L = S^1$

$\mathbb{C} \setminus L$

- $p(z) - 1 \quad n$ distinct roots

maybe



L is compact

$\mathbb{C} \setminus L$ open $= U_\infty \cup \bigcup_i U_i$

$\gamma_i = \partial \overline{U_i}$ $\quad \gamma_i \subset L$

$|p(z)| = 1$ on γ_i

<u>Lemma</u> p vanishes on U_i

p has at most n roots

$\# i\text{'s} \leq n$

$\Rightarrow \#$ components $\leq n+1$

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha \frac{dx}{x-a}$$

$t = \dfrac{x}{1-x}$, $\quad (1-x)t = x$, $\quad t = x(1+t)$

$$dt = dx \frac{1-x+x}{(1-x)^2} = \frac{dx}{(1-x)^2}$$

$$x = 0 \longleftrightarrow t = 0$$
$$x = 1 \longleftrightarrow t = +\infty$$

$$I = \int_0^\infty t^\alpha \frac{dt}{(1+t)^2 \left(\frac{a+(1-a)t}{(1+t)}\right)}$$

$$t = x(1+t)$$

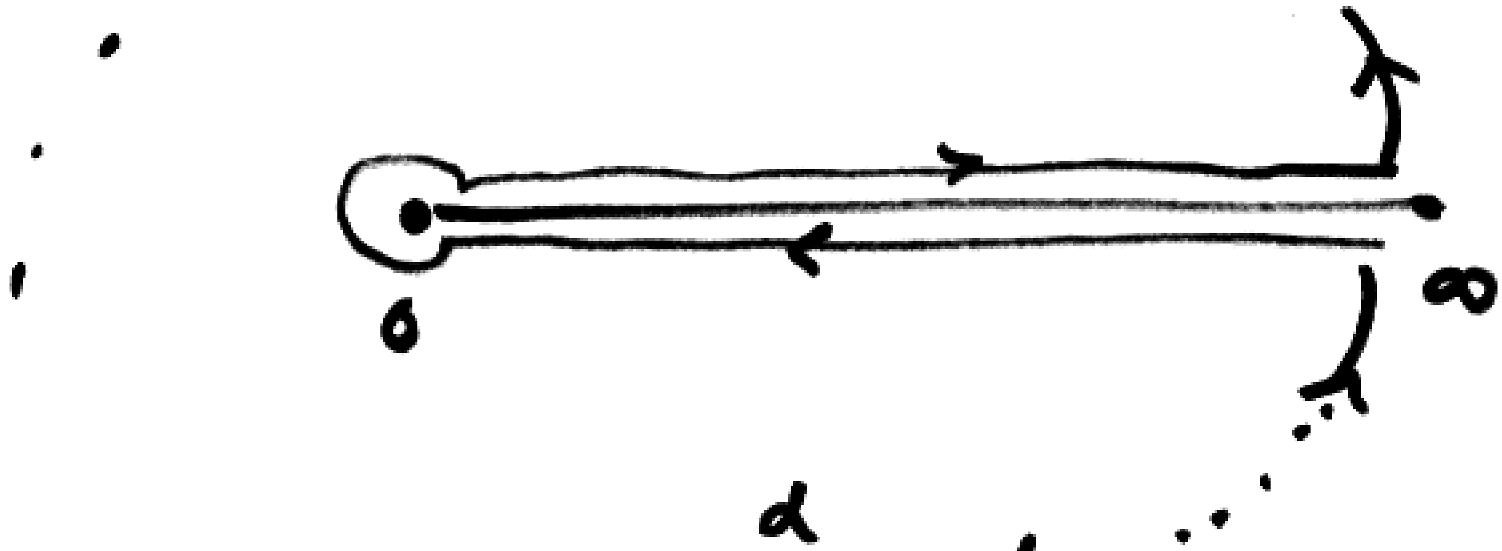
$$\frac{t}{1+t} = x \qquad 1 - x = 1 - \frac{t}{1+t}$$
$$= \frac{1}{1+t}$$

$$dx = (1-x)^2 dt$$
$$= \frac{dt}{(1+t)^2}$$

$$x - a = \frac{t}{1+t} - a = \frac{-a+(1-a)t}{1+t}$$

$$I = \int_0^\infty t^\alpha \frac{dt}{(1+t)\left(-a+(1-a)t\right)}$$

finite $\alpha > -1$



$$f(z) = z^{\alpha} \frac{1}{(1+z)(-a+(1-a)z)}$$

z^{α} defined in $\mathbb{C} \setminus [0, \infty]$

use $0 < \arg(z) < 2\pi$

$$\int_0^{\infty} t^{\alpha} R(t)\, dt$$

R no poles on $(0, \infty)$

R

locally around

$$\int_0^{*} \frac{1}{x^{\beta}}\, dx$$

finite iff $\beta < 1$

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$\beta \geq 0$

$\beta \neq 1$

$$\int x^{-\beta} dx = \frac{1}{-\beta+1} x^{-\beta+1}$$

$$1-\beta > 0$$

$$1 > \beta$$

$$\int_0^{*} \frac{f(x)}{x^\beta} dx$$

$f \neq 0, \infty$
at $x=0$

$$\int_{*}^{\infty} \frac{1}{x^\beta} dx$$

$$\int x^{-\beta} dx = \frac{1}{-\beta+1} x^{-\beta+1}$$

$x = N$ $\qquad \frac{1}{-\beta+1} N^{1-\beta}$

$$1-\beta < 0$$

$$1 < \beta$$

$$\int_{*}^{\infty} \frac{f(x)}{x^\beta} dx$$

$f \to C \neq 0, \infty$
as $x \to \infty$

$$\int_0^\infty t^\alpha \frac{dt}{(1+t)(-a+(1-a)t)}$$

$$\alpha - 2 < -1$$
$$\alpha < 1$$

$$\int_0^\infty t^\alpha . \log t . R(t)\, dt$$

$$\int_0^* t^\alpha \log t \, dt \qquad \alpha > -1$$

$$I(\alpha) = \int_0^\infty t^\alpha R(t)\, dt$$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^\infty t^\alpha . \log t . R(t)\, dt$$

$$(e^{\log t \alpha})' = e^{\log t \alpha} . \log t.$$