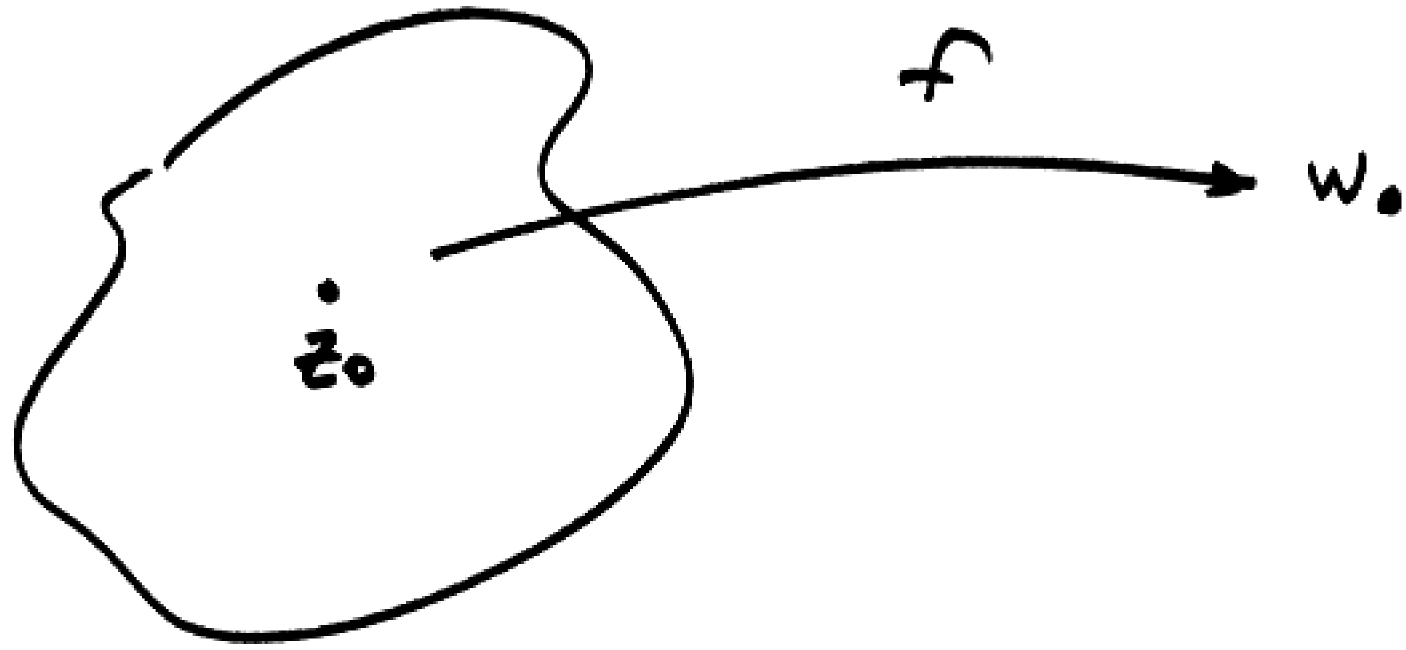
March 1, 2006

f analytic domain U



$$f(z) - f(z_0) = (z - z_0)^n f_n(z)$$

- f_n analytic
- $f_n(z_0) \neq 0$

$f(z) = w_0$ multiplicity n

Shrink disk about z_0 s.t. can take an n^{th} root of f_n

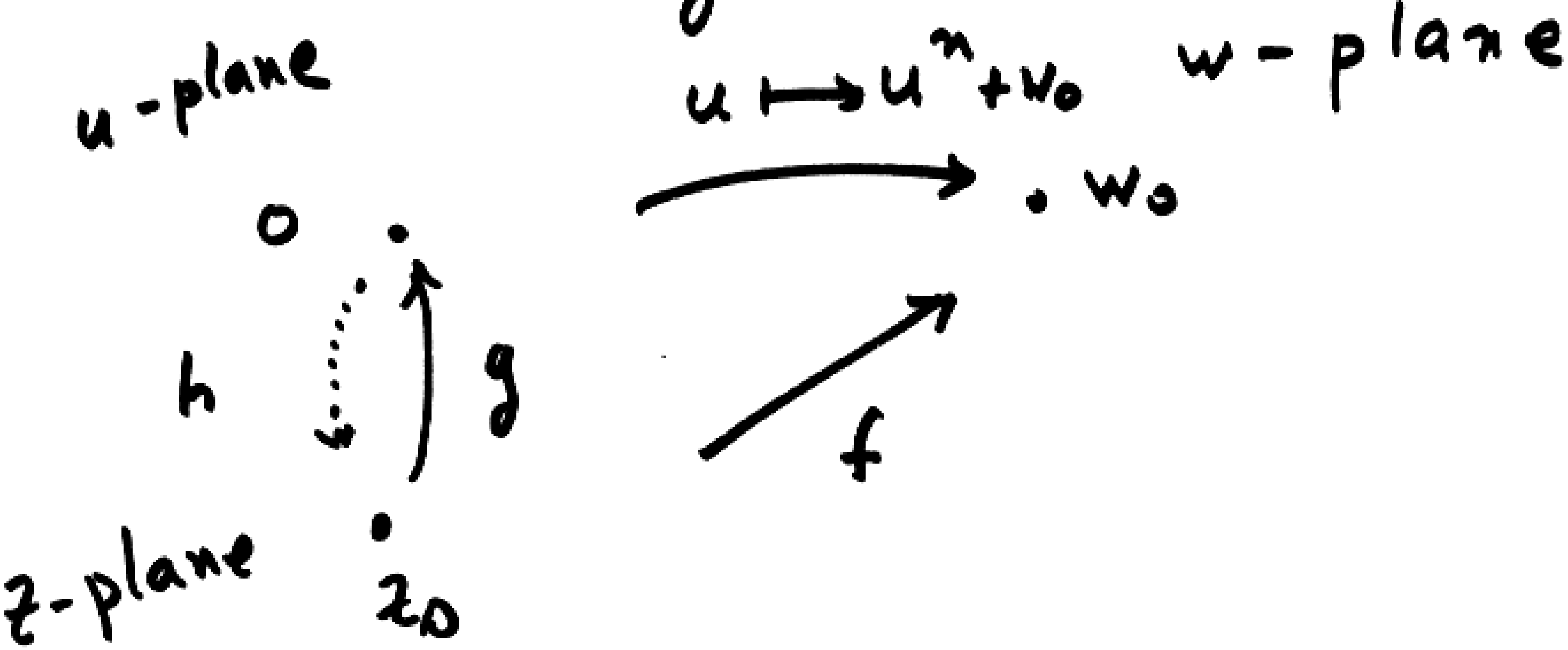
$$f(z) - f(z_0) = \Big(\underbrace{(z - z_0) \cdot f_n^{1/n}(z)}_{g}\Big)^n$$

$g(z_0) = 0$ multiplicity 1

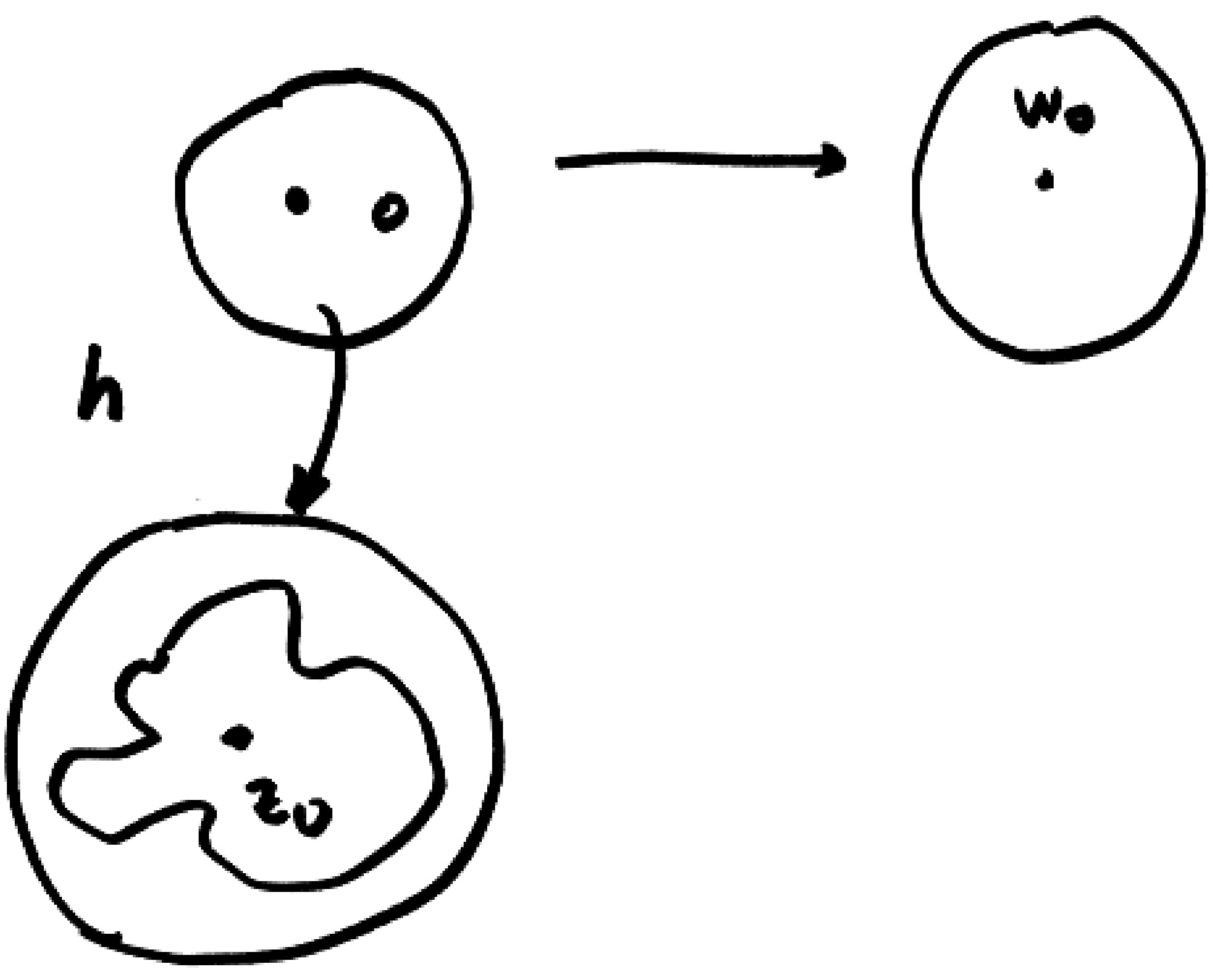
$$w = f(z)$$

$$w - w_0 = u^n$$

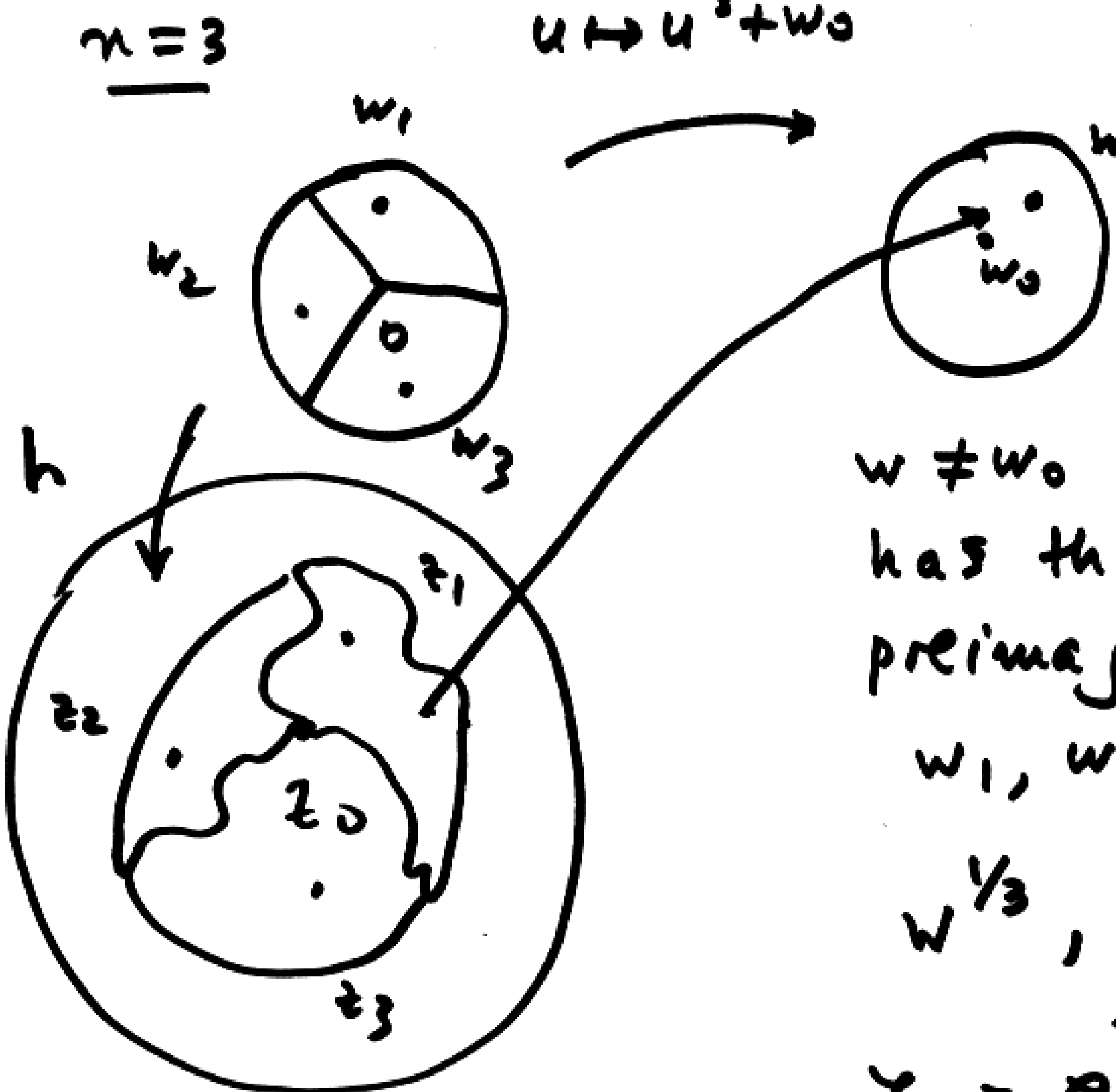
$$u = g(z)$$



g has locally an inverse h



$\underline{n=3}$



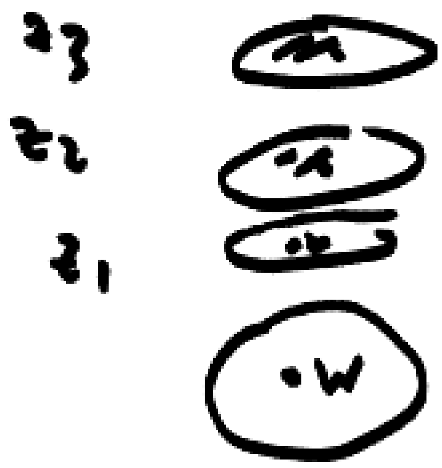
$w \neq w_0$ has three distinct preimages

w_1, w_2, w_3

$w^{1/3}, \zeta_3 w^{1/3}, \zeta_3^2 w^{1/3}$

$\zeta_3 = e^{2\pi i/3}$

$z_i = h(w_i)$



covering.

$w \neq w_0$

We can't have an inverse at a point with multiplicity > 1.

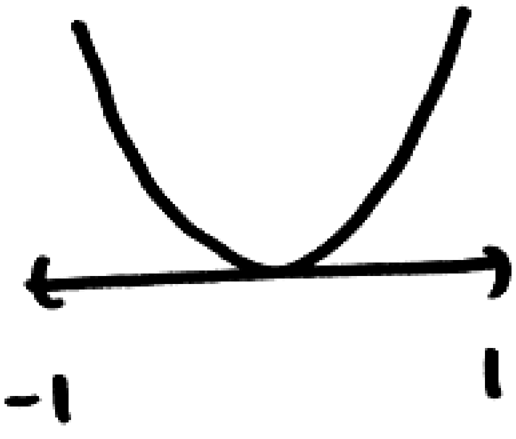
- $f : U \to \mathbb{C}$ analytic

non-constant $\quad U$ domain

f takes open sets to open sets.

$\sin x$

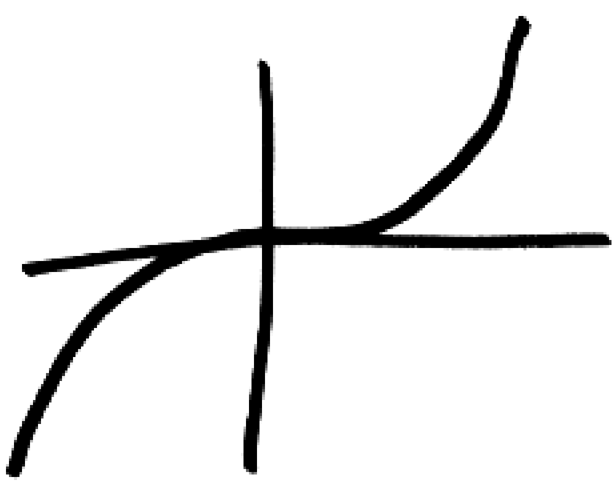
x^2



takes $(-1, 1)$ to $[0,1)$

- For real functions

e.g. $f(x) = x^3$ multiplicity 3 at 0



which has inverse $x^{1/3}$

Let $V \subseteq U$ be an open set

$z_0 \in V \qquad w_0 = f(z_0)$

we can find a nbhd of w_0 and one of z_0 s.t. $f(z) = w$

has n- solutions z, w in the corresponding nbhds.

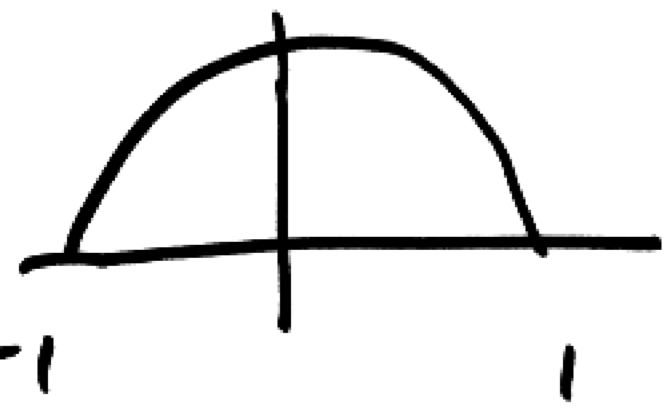
i.e. $w_0 \in V$ has a nbhd entirely contained in V.

- <u>Maximum</u> <u>principle</u>

$f : U \to \mathbb{C}$ $\quad$ f analytic, U domain

then if $|f(z)|$ attains its maximum on U then f is constant.

$1 - x^2$



By open mapping theorem

V = image of U by f

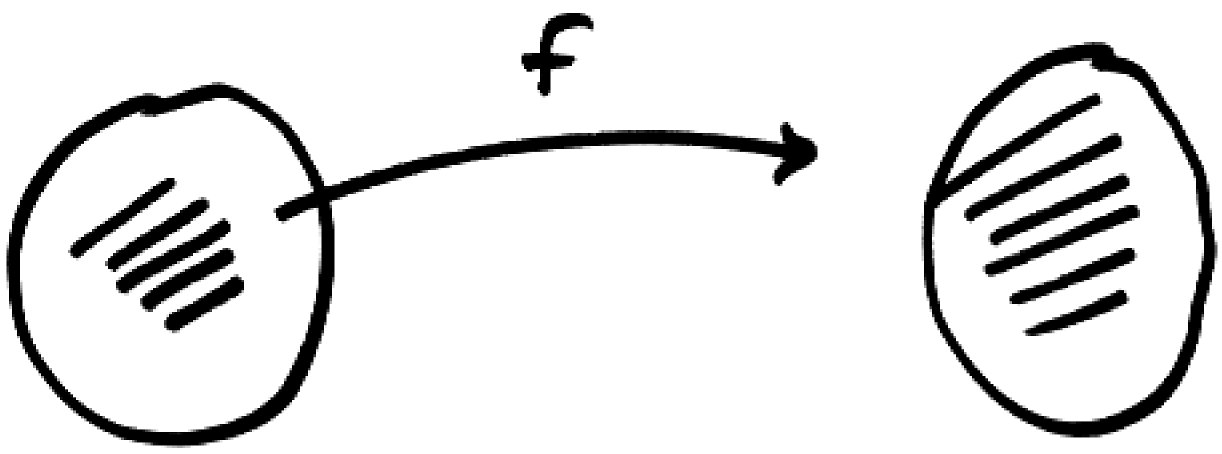
z_0

This disk contains points with larger absolute value.

(Another pf using Cauchy's fmla)

Schwarz Lemma

f analytic $|z|<1$



unit disk   unit disk

$$|f(z)| \le 1 \quad \text{for} \quad |z|<1$$
$$f(0)=0$$

Then

$$|f(z)| \le |z|$$

and $|f'(0)| \le 1$

Pf

$$f_1(z) := \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

is analytic in $|z|<1$

want to prove

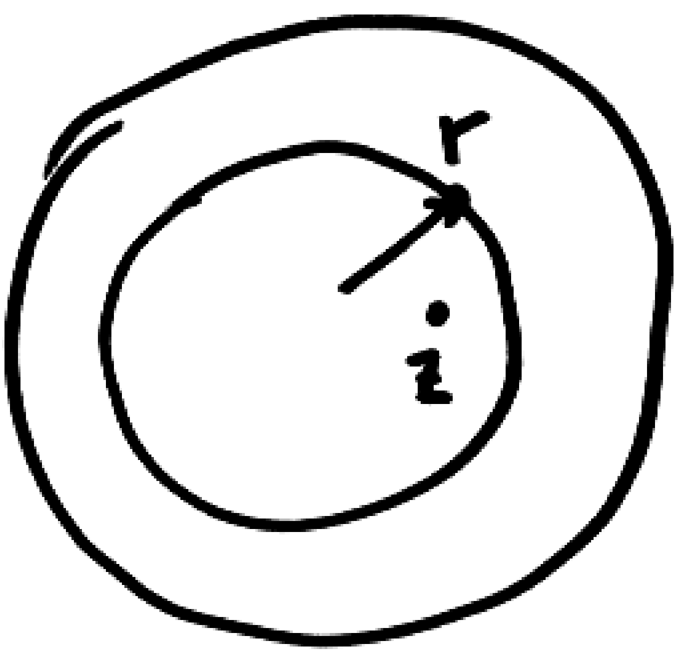
$$|f_1(z)| \leq 1$$

take $\quad |z| \leq r < 1$

$|f_1(z)|$ takes its maximum on boundary

$$|f_1(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \qquad |z| \leq r$$

$r \to 1 \qquad |f_1(z)| \leq 1.$



$$\Rightarrow \quad |f(z)| \leq |z|$$

$$\& \quad |f'(0)| \leq 1 \qquad \square$$

pblm

f entire

$$M(r) := \max_{|z|=r} |f(z)|$$

If $\dfrac{M(r)}{r^n} \to 0$ as $r \to \infty$

for some n then

f is a polynomial of $\deg < n$

Cauchy fmla

$$\frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{m+1}}\, dz$$

$m \geqslant n$ $$\leq \frac{M(r)}{r^m} = \frac{M(r)}{r^n} \frac{1}{r^{m-n}}$$

let $r \to \infty$ $$\frac{f^{(m)}(0)}{m!} = 0$$

since f is entire it equals ($m \geqslant n$)

its Taylor series which

such a polynomial.

March 3, 2006

f, g entire no common zeros

$\exists$ a, b entire

$$af + bg = 1$$

$$\frac{a}{g} + \frac{b}{f} = \frac{1}{f \cdot g}$$

Consider f, g polynomials

partial fraction expansion

$$\frac{1}{f \cdot g} = \sum_{\nu} h_\nu$$

h_ν has only one pole

i.e. $\dfrac{\alpha_K}{(z - a_\nu)^K}$ α_K polynomial of deg $< K$

or a polynomial.

Because f & g have no common zeros we can write the sum as

$$\sum_{\nu} f_\nu + \left(\sum_{\mu} g_\mu + \text{polynomial} \right)$$

poles of f_ν $\longleftrightarrow$ zeros of f

poles of g_μ $\longleftrightarrow$ zeros of g

$$\sum_\nu f_\nu = \frac{\tilde{a}}{f}$$

$\tilde{a}$, b polynomials

$$\text{polyn} + \sum_\mu h_\mu = \frac{\tilde{b}}{g}$$

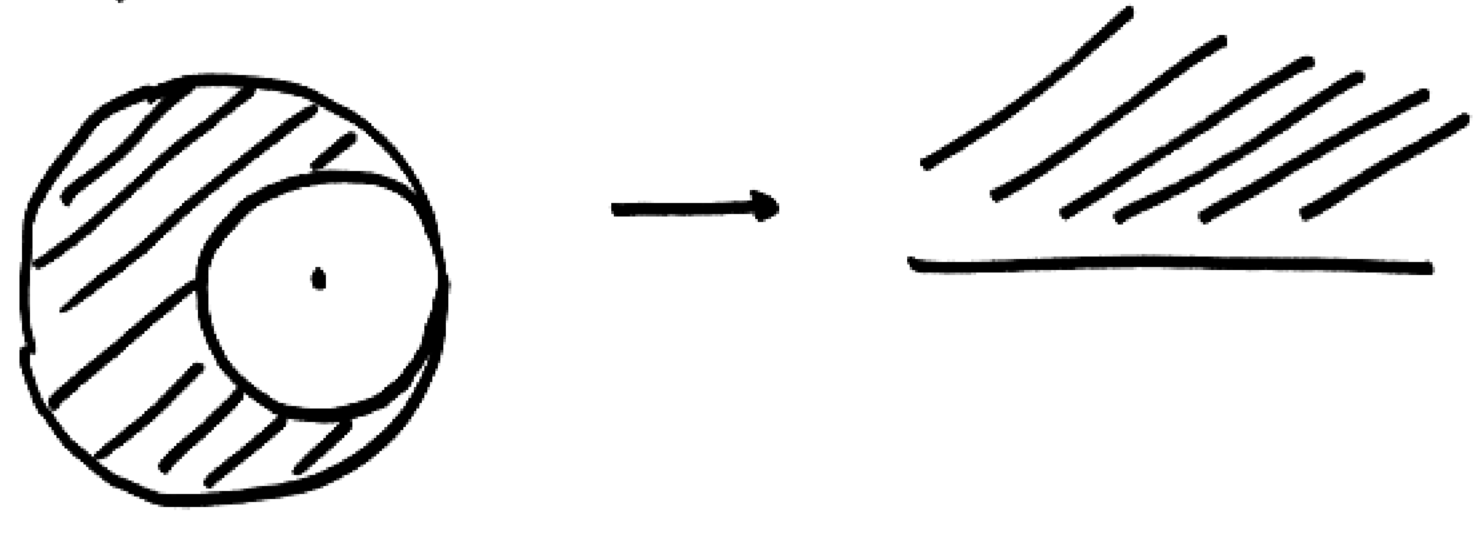
$$\frac{\tilde{a}}{f} + \frac{\tilde{b}}{g} = \frac{1}{fg}$$

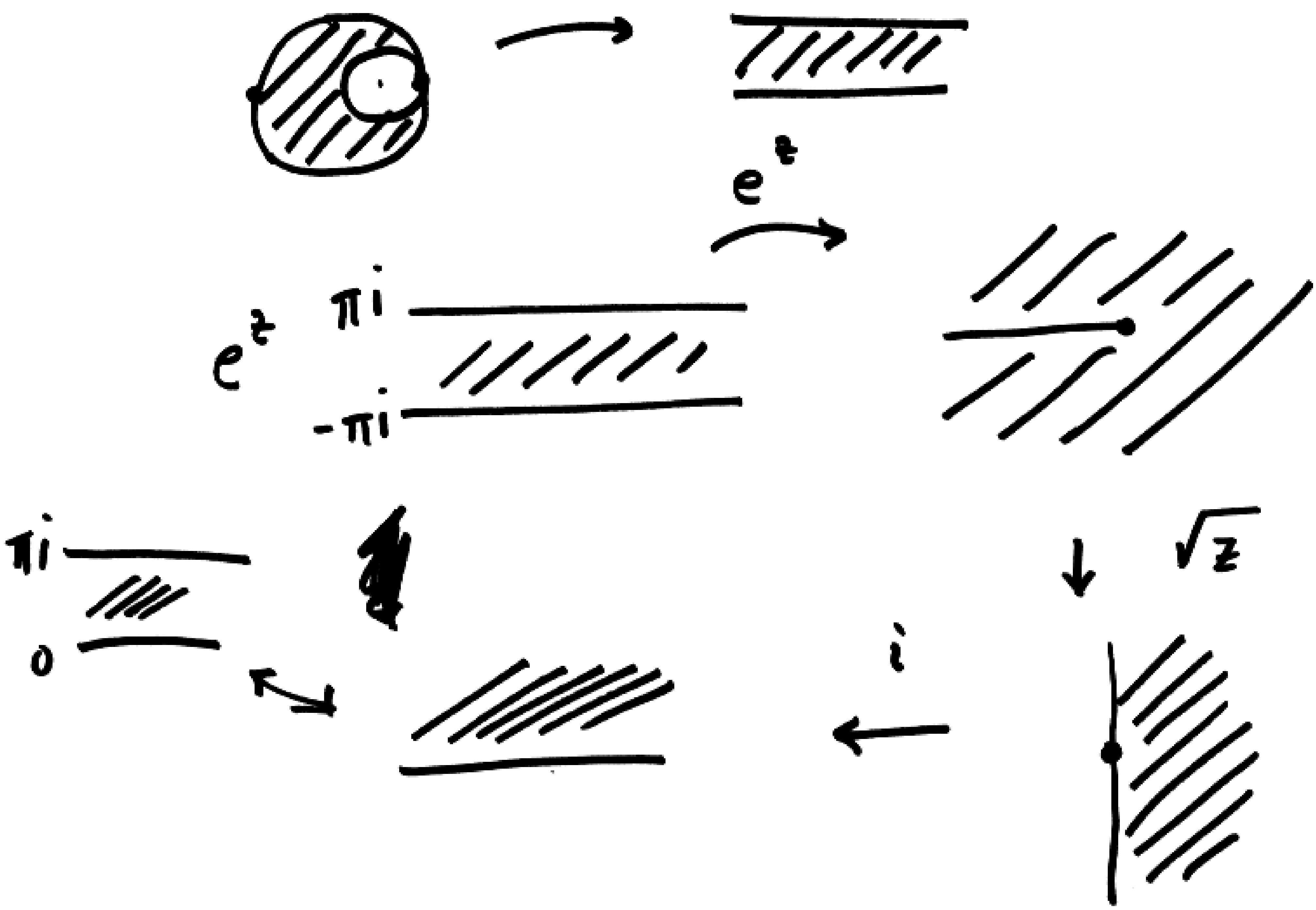
$$\Rightarrow \quad ag + bf = 1$$

$$\{ z \mid \ |z| < 2 \quad |z-1| > 1 \}$$

$\downarrow$ conformal homeo

$$\{ z \mid \operatorname{Im} z > 0 \}$$





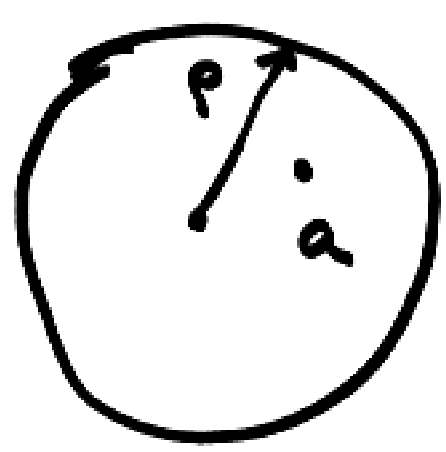
$$\frac{1}{i}\,\frac{z+2}{z-2}$$

$$z = 2i$$

$$\frac{2+i2}{2i-2} = \frac{1+i}{i-1} = -i$$

$$\frac{az+b}{z-2}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^4}$$



$|z|=1$

$$|z-a|^4 = (z-a)^2(\bar{z}-\bar{a})^2$$
$$= (z-a)^2(z^{-1}-\bar{a})^2$$

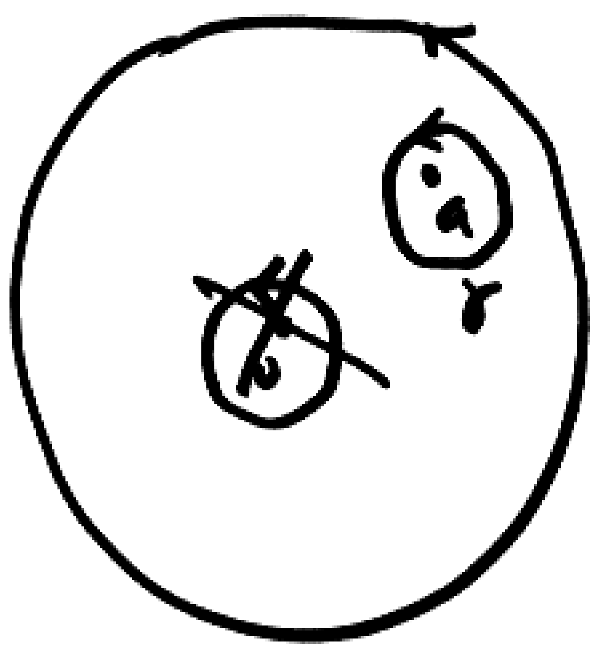
$$|dz| = -i\,\frac{dz}{z}$$

$$z = e^{i\theta}$$
$$dz = i e^{i\theta} d\theta$$

$$-\frac{dz}{iz} = d\theta$$

$$\frac{-1}{i}\int_{|z|=1} \frac{dz}{z(z-a)^2(z^{-1}-\bar{a})^2}$$

$$1-\bar{a}z \qquad \frac{1}{\bar{a}} = \frac{a}{|a|^2}$$



$$\frac{1}{2\pi i}\int_{\gamma} \frac{z\,dz}{(z-a)^2(1-\bar{a}z)^2}$$

$$f(z) = \frac{z}{(1-\bar{a}z)^2}$$

analytic at a

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{(z-a)^2}$$

$$= \frac{f'(a)}{1!}$$

similarly with a outside

$\rightarrow$ $1/\bar{a}$ is inside ...

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$z = e^{i\theta}$ $$= \frac{z + z^{-1}}{2}$$

$$= \frac{z - z^{-1}}{2i}$$

$$R(\cos\theta, \sin\theta) \rightsquigarrow R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$$

f entire

For every $a \in \mathbb{C}$ the power series expansion of f at a has at least one zero coeff.

$\Rightarrow$ f is a polynomial.

$$\mathbb{C} = \bigcup_{n \geq 0} \{ z \mid f^{(n)}(z) = 0 \}$$

one of them is uncountable say $f^{(n)}$

hence $f^{(n)} \equiv 0$