Cplx Analysis SPRING '06

MIDTERM

Name: ______________________

Please give your answers in excruciating detail.

1. Let H be the ring of entire functions.

(a) Characterize the units of H and show there are infinitely many. non-constant units.

(b) Show that H is an integral domain (i.e., $f \cdot g = 0 \Rightarrow f = 0$ or $g = 0$)

(c) For $a \in \mathbb{C}$, prove $z - a \in H$ is irreducible (h is irreducible iff $h = f \cdot g \Rightarrow f$ or g are units)

(d) Show any irreducible of H is of the form $u(z-a)$ for some $a \in \mathbb{C}$ and some unit u in H.

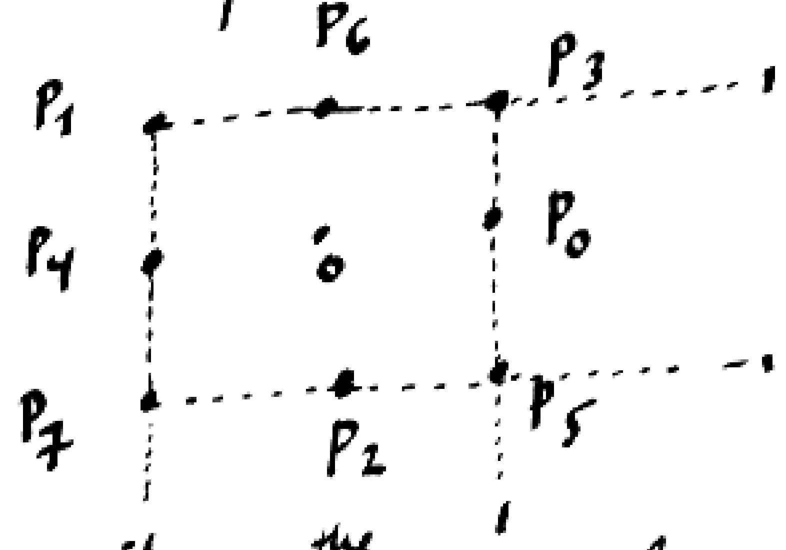
(e) Prove that not every $h \in H$ is a finite product $h = h_1 \cdots h_n$ of irreducibles h_i.

2. Let U be a domain and $f: U \to \mathbb{C}$ an analytic function. Prove that if f is injective then it has an analytic inverse

$$g: f(U) \to \mathbb{C}$$

$$g \circ f(z) = z$$

3. Consider the following points P_i on the unit square $i=0,1,\dots,7$



Let γ be the path formed of segments

$$\overline{P_0P_1}\,,\ \overline{P_1P_2}\,,\ \dots\,,\ \overline{P_6P_7}\,,\ \overline{P_7P_0}$$

in this order.

Compute

$$\int_\gamma \frac{\cos z \cdot e^{z^2}}{\sin z \cdot (z-2)}\, dz$$

4. Let $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ be a power series with radius of convergence $R>0$. Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$ be the power series with coefficients, given recursively by

$n \geq 1$, $\quad c_n = -(a_n + a_{n-1}c_1 + \dots + a_1 c_{n-1})$

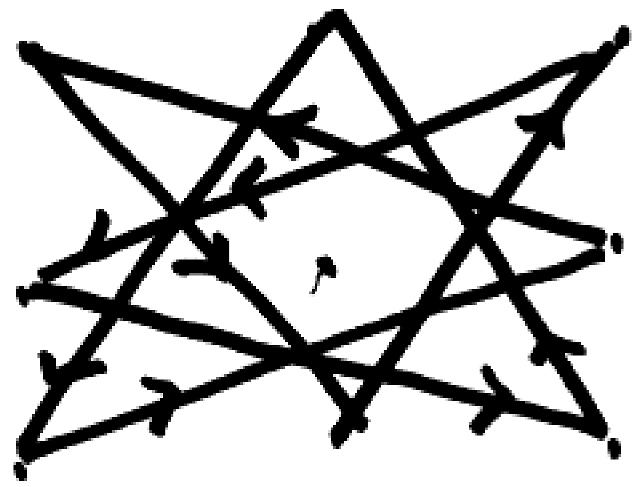
Let $\mu(r) := \sup_{|z|=r} |f(z)|$ for $r<R$.

(a) Show that $f(z)\cdot h(z) = 1$ as formal power series.

(b) Show that h has radius of convergence at least $\dfrac{r}{1+\mu(r)}$ (HINT: Show $|c_K| \le \frac{\mu(r)}{r}\left(\frac{1+\mu(r)}{r}\right)^{K-}$ $R>r>0$

March 8, 2006

3.



γ

$$\int_\gamma \frac{\cos z \cdot e^{z^2}}{\sin z \, (z-2)} \, dz$$

$\sin 0 = 0$

$\sin(\pm \pi) = 0$

$$f(z) = \frac{\cos z \cdot e^{z^2}}{\frac{\sin z}{z}(z-2)}$$

$$\int_\gamma \frac{f(z)}{z} \, dz = 3 \times 2\pi i \times f(0)$$

$$= 6\pi i \cdot \frac{1}{-2} = -3\pi i$$

2

4. $f(z) = \overset{a_0 =}{1} + a_1 z + a_2 z^2 + \dots$

$R > 0$

$$\frac{1}{f(z)} = \underset{= c_0}{1} + c_1 z + c_2 z^2 + \dots$$

$$1 = (1 + a_1 z + a_2 z^2 + \dots)(1 + c_1 z + c_2 z^2 + \dots)$$

$$\sum_{k=0}^{n} a_{n-k}\, c_k = 0 \quad \text{for } n \geq 1$$

$$c_n = -(a_n + a_{n-1} c_1 + \dots + a_1 c_{n-1})$$

$$\mu(r) := \sup_{|z|=r} |f(z)| \qquad 0 < r < R$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz$$

$$|a_n| \leq \mu(r) \frac{1}{r^n}$$

Assume

$$|c_k| \le \frac{\mu(r)}{r}\left(1+\frac{\mu(r)}{r}\right)^{k-1}$$

$$|c_n| \le \frac{\mu(r)}{r^n} + \sum_{k=1}^{n-1} \frac{\mu(r)}{r^{n-k}}\cdot\frac{\mu(r)}{r}\left(1+\frac{\mu(r)}{r}\right)^{k-1}$$

$$= \frac{\mu(r)}{r^n} + \sum_{k=1}^{n-1} \frac{\mu(r)^2}{r^n}\cdot\left(1+\mu(r)\right)^{k-1}$$

$$= \frac{\mu(r)}{r^n}\left[1+\mu(r)\,\frac{(1+\mu(r))^{n-1}-1}{1+\mu(r)-1}\right]$$

$$= \frac{\mu(r)}{r^n}\left(1+\mu(r)\right)^{n-1}$$

$$= \frac{\mu(r)}{r}\left(1+\frac{\mu(r)}{r}\right)^{n-1}$$

$\underline{K=1}$ $\quad \underbrace{|c_1|}_{=\,|a_1|} \le \dfrac{\mu(r)}{r}$

$$\frac{1}{1\mp x} = 1 \pm x + O(x^2)$$

Proved the hint

$$|C_k| \leq \frac{\mu(r)}{r}\left(\frac{1+\mu(r)}{r}\right)^{k-1}$$

$$\frac{1}{R} = \limsup_{k\to\infty} |C_k|^{1/k} \leq \frac{1+\mu(r)}{r}$$

$$R \geq \frac{r}{1+\mu(r)} \qquad 0 < r < R$$

$$\boxed{R \geq \sup_{0<r<R} \frac{r}{1+\mu(r)}} \quad > 0$$

$$f(z) = \frac{1}{z-1}$$

$$\frac{1}{f(z)} = z - 1 \qquad R = \infty$$

2. $f : U \to \mathbb{C}$

analytic injective

Prove it has analytic inverse

$g : f(U) \to \mathbb{C}$

$g \circ f(z) = z$

If $f'(z_0) = 0$ then f cannot be injective on a nbhd of z_0

$f'(z_0) = 0 \Longrightarrow f(z) - \cancel{z_0}\, f(z_0)$ has a zero of order $\underset{= n}{\text{order}}$ at least 2

$f(z) = w_0 \qquad w_0 \neq z_0$

n preimages

1. (a) u entire fctn

unit $\quad u\,v = 1$

for some v entire.

$\Leftrightarrow$ u does not vanish in $\mathbb{C}$

$e^z, \quad e^{f(z)} \quad \ldots$

(b) $\quad f \cdot g = 0 \Rightarrow \begin{matrix} f = 0 \\ g = 0 \end{matrix}$ OR

$f(z_0) \neq 0 \Rightarrow f(z) \neq 0$ on disk about z_0

$\Rightarrow g = 0$ on disk

$\Rightarrow g = 0$ in $\mathbb{C}$.

(c) $\quad z - a = f \cdot g$

either f or g are units.

(d)

(e). $\quad h_1 \cdots h_n = h \qquad e^z - 1$

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$$f(z) = 1 + a_1 z + a_2 z^2 + \dots$$

Radius of convergence ≥ 1

$$|a_n| \leq 1$$

$f(z) \neq 0$ for $|z| \leq 0.1715\dots$

If $f(z) \neq 0$ on $|z| \leq r$

then $\dfrac{1}{f(z)}$ is analytic there

hence the radius of convergence of the series

$$g(z) = \frac{1}{f(z)} = 1 + c_1 z + c_2 z^2 + \dots$$

is at least r

$R :=$ Radius of convergence of g is

$$\geq \frac{r}{1+\mu(r)}, \quad \mu(r) = \max_{|z|=r} |f(z)|$$

$|z| = r$

$$|f(z)| \le 1 + |a_1| r + |a_2| r^2 + \dots$$

$$\le \frac{1}{1-r} \qquad r < 1$$

$$\Rightarrow \quad M(r) \le \frac{1}{1-r}$$

$0 < r < 1$

$$R \geq \frac{r}{1 + \frac{1}{1-r}}$$

$$= \frac{r(1-r)}{2-r} = \frac{(r-1)r}{(r-2)}$$

$$\text{deriv} = \frac{r^2 - 4r + 2}{(r-2)^2}$$

$$\frac{(2r-1)(r-2) - r(r-1)}{(r-2)^2}$$

max occurs at root of $r^2 - 4r + 2$ in $[0, 1]$ $r = 0.5857\dots$

max $= 0.171528\dots$

polynomials? $1 + a_1 z$

zero at $z = -\frac{1}{a_1}$.

—m—

$$f(z) = z e^z = z + z^2 + \frac{z^3}{2} + \cdots$$

$$g(z) = z + b_2 z^2 + \cdots$$

$$f \circ g(z) = z$$

Identify b_n?

$$b_n = \frac{(-n)^{n-1}}{n!}$$

$$f = a_1 z + a_2 z^2 + \cdots, \quad a_1 \neq 0$$

$$f_1(z) = a_1 + a_2 z + \cdots = \frac{f(z)}{z}$$

$$\frac{1}{f_1(z)} = c_0 + c_1 z + \cdots$$ radius of convergence $R > 0$

Cauchy estimates

$$|c_n| \leq \frac{M_1(r)}{r^n} \qquad n = 0, 1, 2 \ldots$$

$$0 < r < R$$

$$\mu_1(r) := \max_{|z|=r} \left| \frac{1}{f_1(z)} \right|$$

$k = 1, 2, \ldots$

$$\frac{1}{f_1(z)^k} = c_0^{(k)} + c_1^{(k)} z + \cdots$$

$$|c_m^{(k)}| \leq \frac{\mu_1(r)^k}{r^m}$$

$$g(z) = b_1 z + b_2 z^2 + \cdots$$

$$f \circ g(z) = z$$

$$\boxed{b_n = \frac{1}{n} c_{n-1}^{(n)}}$$

E.g.

$$f(z) = z e^z$$

$$f_1(z) = e^z$$

$$\frac{1}{f_1(z)^k} = e^{-kz} = 1 - \frac{k}{1!} z + \frac{k^2}{2!} z^2 + \cdots$$

$$C_n^{(k)} = \frac{(-k)^n}{n!}$$

$$b_n = \frac{1}{n} C_{n-1}^{(n)} = \frac{1}{n} \frac{(-n)^{n-1}}{(n-1)!}$$

$$= \frac{(-n)^{n-1}}{n!}$$

Apply to our bound

$$|b_n| \leq \frac{1}{n} \frac{\mu_1(r)^n}{r^{n-1}}$$

$$\rho^{-1} = \limsup_{n \to \infty} |b_n|^{1/n} \leq \frac{\mu_1(r)}{r}$$

$\rho :=$ radius of convergence of g

$$\Rightarrow \quad \rho \geq \frac{r}{\mu_1(r)} > 0$$

$$\mu_1(r) = \max_{|z|=r} \left| \frac{z}{f(z)} \right|$$

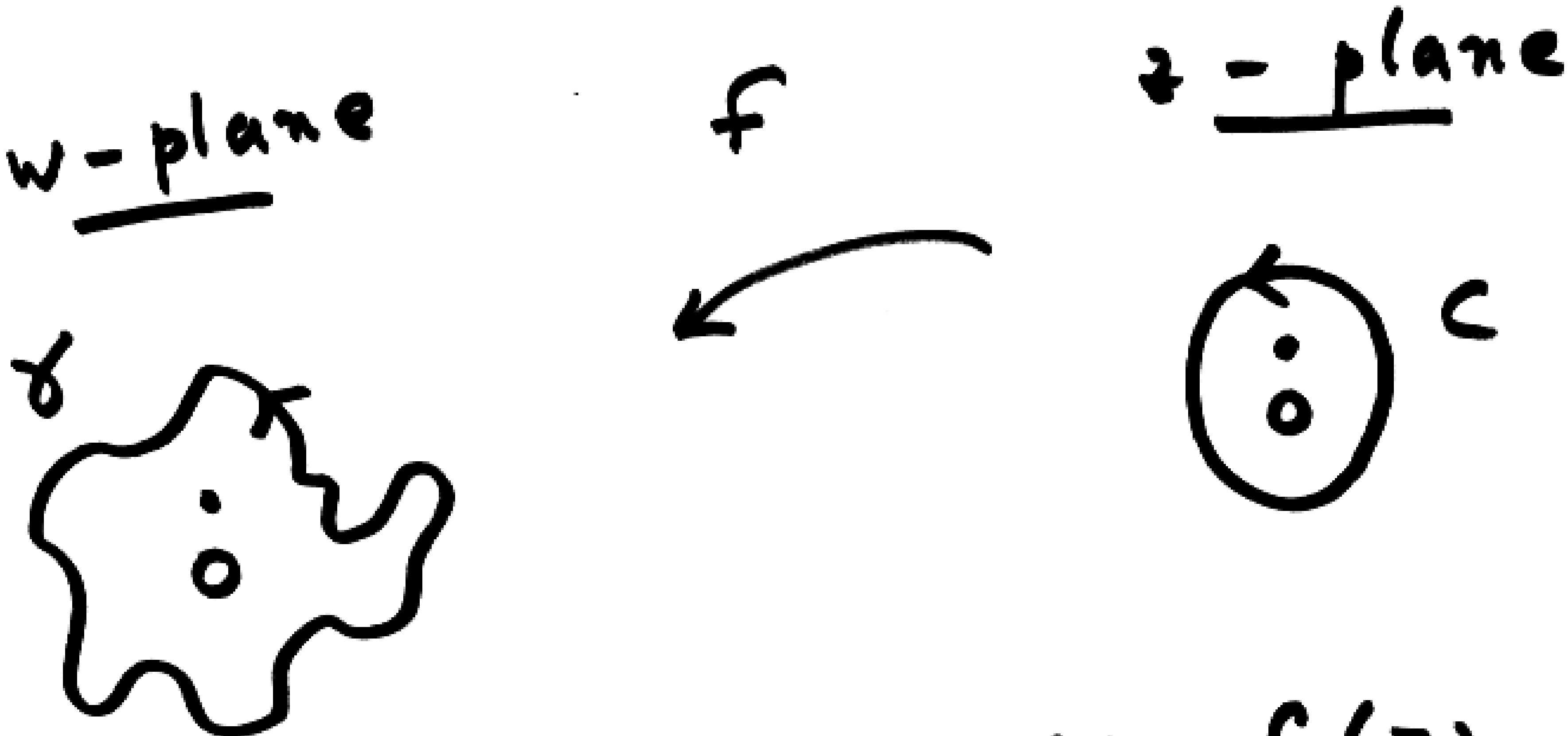
Proof of claim

Find coeff of $z \cdot g'(z) =: h(z)$

by Cauchy's fmla.

n^{th} coeff $=$ $n b_n$

$$n b_n = \frac{1}{2\pi i} \int_\gamma \frac{h(w)\, dw}{w^{n+1}}$$



$w = f(z)$

$$\gamma(t) = f(C(t))$$

$$\gamma'(t) = f'(C(t)) \cdot C'(t)$$

$$n b_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{h(\gamma(t))}{\gamma(t)^{n+1}} \gamma'(t)\, dt$$

$$h(\gamma(t)) = \gamma(t)\, g'(\gamma(t))$$

$$= f(c(t)) \cdot g'(f(c(t)))$$

$$= \frac{f(c(t))}{f'(c(t))}$$

$$\rightarrow \quad n b_n = \frac{1}{2\pi i} \int_C \frac{1}{f(z)^n}\, dz$$

$$= \sum_{m \geq 0} C_m^{(n)} \underbrace{\frac{1}{2\pi i} \int_C z^{m-n+1} \frac{dz}{z}}_{\substack{= 0 \\ \text{unless } m-n+1=0}}$$

$$= C_{n-1}^{n} \qquad \square$$