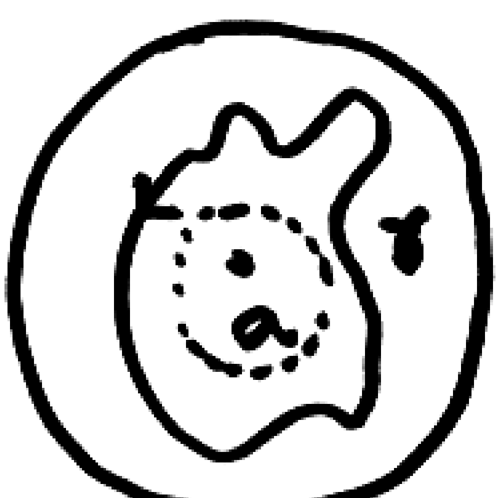
Residues

f analytic on a disk and γ is closed path then

$$\int_\gamma f(z)\,dz = 0$$

f analytic on a disk centered at a but not necessarily at a.



$$\alpha := \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$

$$n(\gamma, a) = 1$$

Residue of f at a

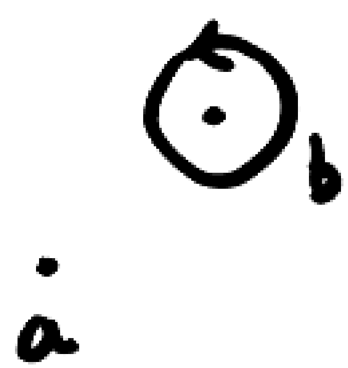
$$\alpha =: \operatorname*{Res}_{z=a} f$$

E.g. $$\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 1$$

$$\operatorname*{Res}_{z=a} \frac{1}{z-a} = 1$$

$$\operatorname*{Res}_{z=b} \frac{1}{z-a} = 0$$

$a \neq b$



E.g $$\operatorname*{Res}_{z=a} \frac{1}{(z-a)^2} = 0$$

$$\frac{1}{2\pi i}\int_C \frac{1}{(z-a)^2}\, dz = 0$$

$$\operatorname*{Res}_{z=a} \frac{1}{(z-a)^k} = 0$$

$k > 1$

$$f(z) = \frac{c_{-k}}{(z-a)^k} + \frac{c_{k-1}}{(z-a)^{k-1}} + \cdots$$

$$+ \frac{c_{-1}}{(z-a)} + g(z)$$

g analytic

f pole of order k at $z=a$

$$\boxed{\operatorname*{Res}_{z=a} f = c_{-1}}$$

If f is ~~meromorphic~~ analytic on a disk about a and has possibly a pole at $z=a$

$$g = \frac{f'}{f}$$

$$\operatorname*{Res}_{z=a} g = k$$

$$f(z) = (z-a)^K h(z)$$

$h(a) \neq 0$ $\quad$ h analytic

$K \in \mathbb{Z}$ $\quad$ order of zero/pole

$$g = \frac{f'}{f} = \frac{K}{z-a} + \frac{h'(z)}{h(z)}$$

$$\operatorname*{Res}_{z=0} \frac{\cos z \cdot e^{z^2}}{\sin z \cdot (z-2)} =: f(z)$$

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \ldots$$

$$c_{-1} = \lim_{z \to 0} z f(z)$$

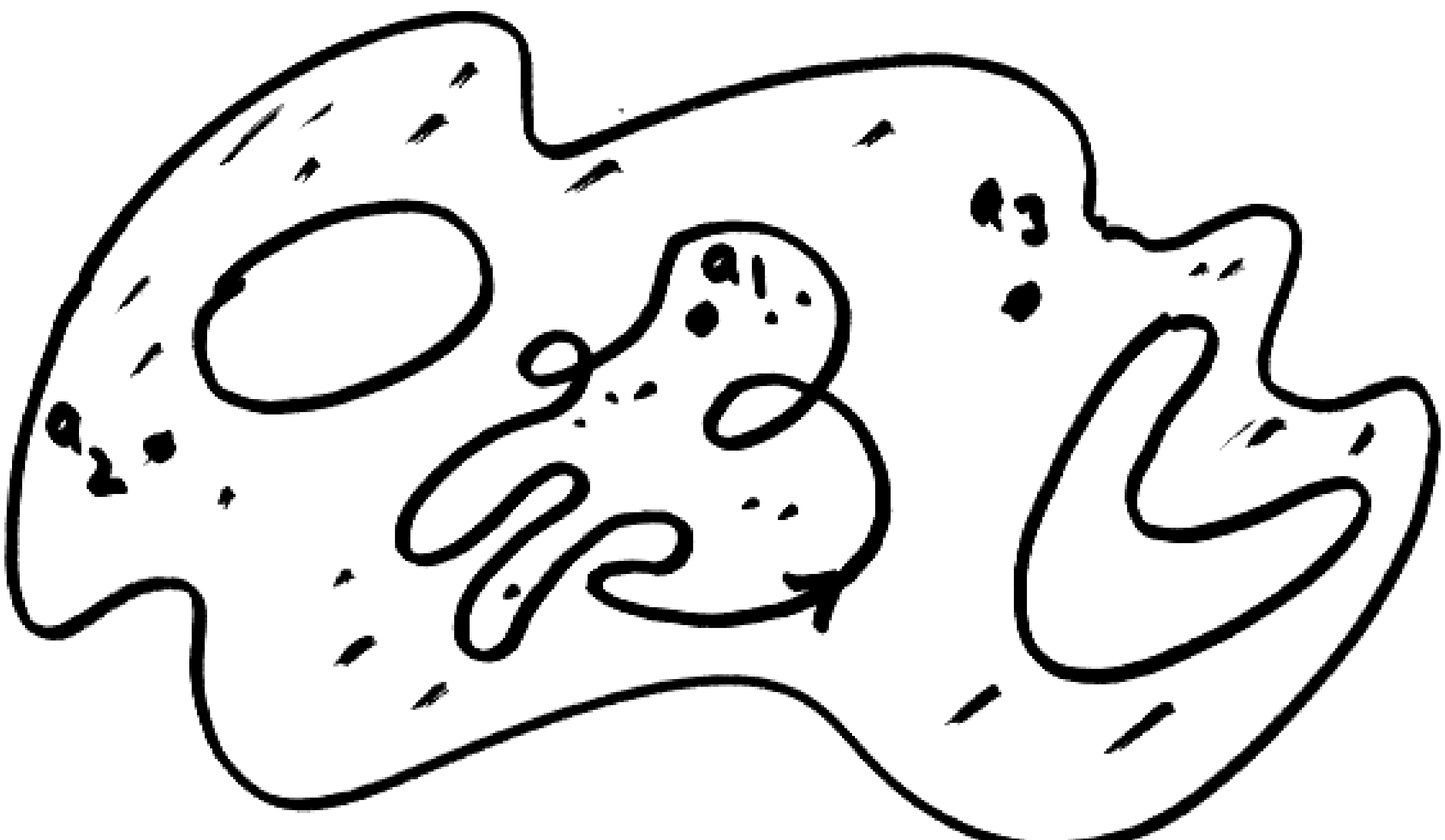
$$= \lim_{z \to 0} \frac{\cos z \cdot e^{z^2}}{\frac{\sin z}{z} \cdot (z-2)} = -\frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2) f(z)$$

$$= \frac{\cos 2 \cdot e^4}{\sin 2}$$

Residue Theorem

U region



γ closed path in U

$n(\gamma, a) = 0 \quad a \notin U$

$\gamma \sim 0$ in U

f analytic in U with (6)
possibly exceptions $a_1, a_2 \ldots, a_N$
(γ not going through the a_i's)

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{N} n(\gamma, a_j)\operatorname*{Res}_{z=a_j} f$$

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f meromorphic

$$\operatorname*{Res}_{z=a} \frac{f'}{f} = \text{order of } f \text{ at } a = k$$

$$f(z) = (z-a)^k g(z)$$

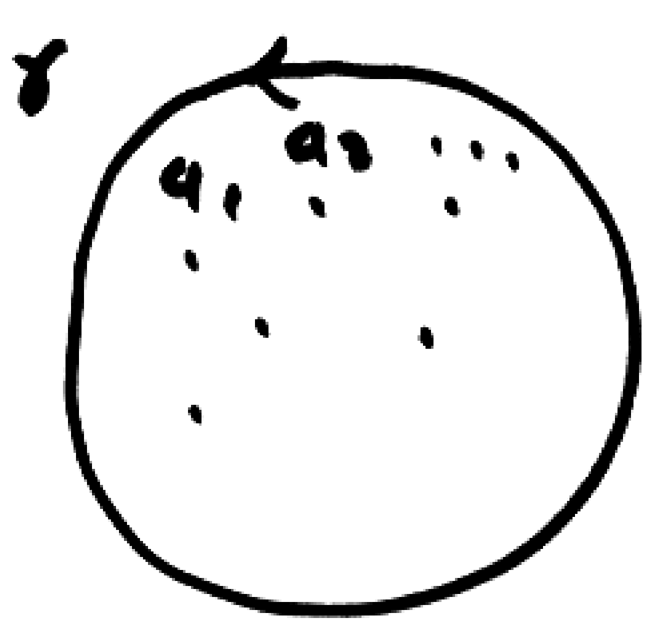
$g(a) \neq 0$
g analytic about a

$k \in \mathbb{Z}$

Apply residue theorem to $\frac{f'}{f}$

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}(z)\, dz = \sum_{j=1}^{n} n(\gamma, a_j) \operatorname*{ord}_{z=a_j} f$$

Typical situation



f meromorphic
disk D
$\gamma = \partial D$

a_j zero/pole of f

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f}(z)\,d(z) = \#\text{zeros of } f - \#\text{poles of } f$$

(counting w/ multiplicity)

<u>Argument principle</u>

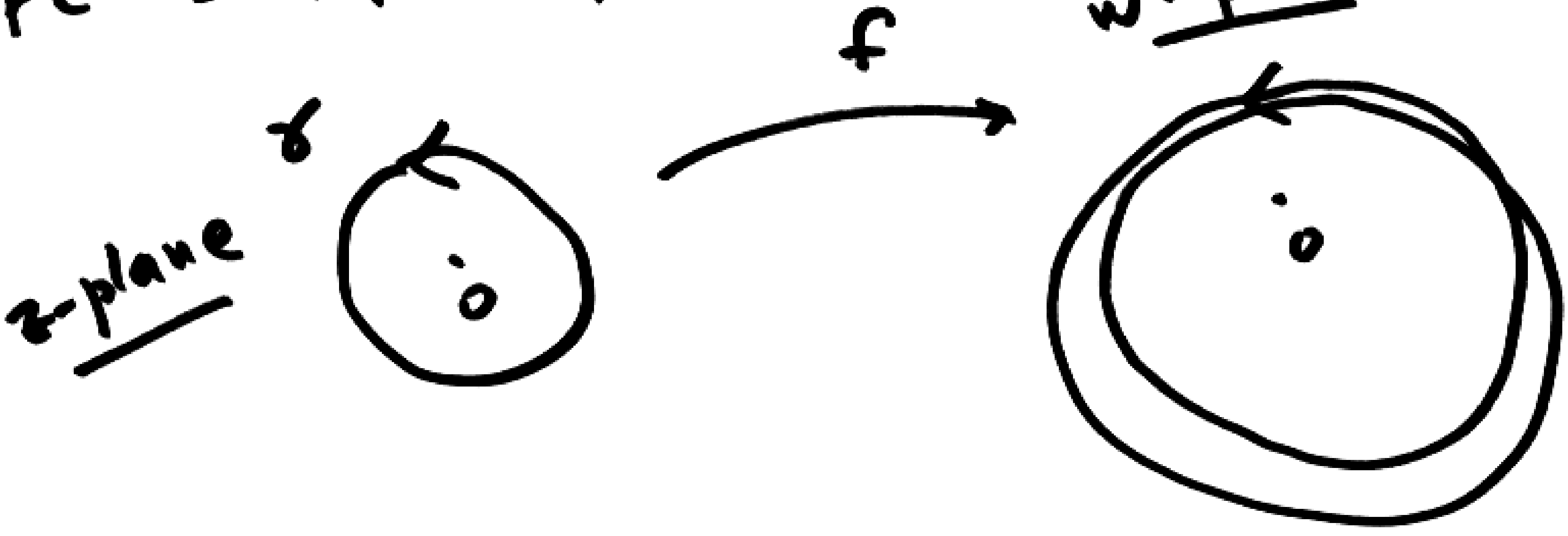
$$f(z) = z^2$$

$$\frac{f'}{f} = \frac{2}{z}$$

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'}{f}(z)\,dz = 2$$

$$z = r\cdot e^{i\theta}$$

$$r^2 e^{2i\theta} = f(e^{i\theta} r)$$



$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\int_\sigma \frac{\sigma'(t)\,dt}{\sigma(t)}$$

$$\sigma = f\circ\gamma$$

$$\sigma' = f'\circ\gamma\cdot\gamma'$$

$$= \frac{1}{2\pi i}\int_\sigma \frac{dw}{w}$$

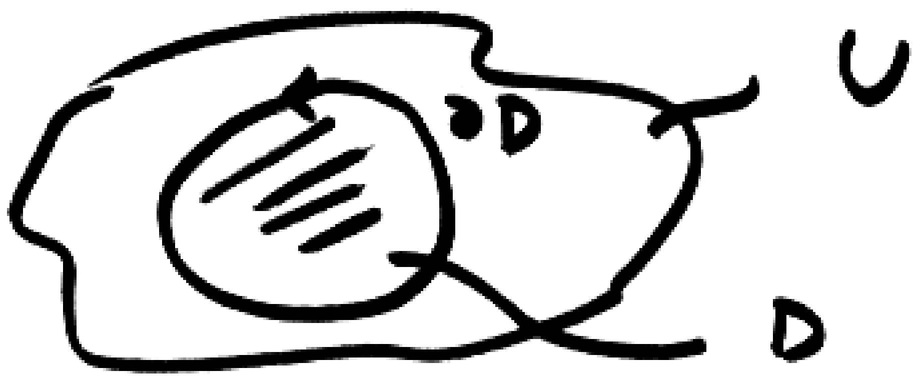
$$= n(\sigma, 0)$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt$$

Rouché's theorem

f, g analytic on $U \supseteq \overline{D}$
disk

If $|f(z) - g(z)| < |f(z)|$
for $z \in \partial D$



then f, g have the same number of zeros on D

<u>Pf</u> $\quad h(z) := \dfrac{g(z)}{f(z)}$

meromorphic on $U \supseteq \overline{D}$

$$\#\text{zeros of } g - \#\text{zeros of } f$$

$$= \#\text{zeros of } h - \#\text{poles of } h$$

$$= \frac{1}{2\pi i} \int \frac{h'}{h}(z)\, dz$$

$$= n(\sigma, 0) \qquad \gamma = \partial D$$

$$\sigma = h \circ \gamma$$

Need to show $n(\sigma, 0) = 0$

$$|1 - h(z)| < 1 \qquad z \in \partial D$$

$$\Rightarrow \sigma : [a,b] \longrightarrow$$ 

$$\Rightarrow \quad n(\sigma, 0) = 0$$

$\square$