• Last time:

C/K curve of genus $g \geq 2$. We constructed an abelian variety $J = \mathrm{Jac}(C)$ also defined over K and of dimension g.

If $P_0 \in C(K)$ then we have a map

$$\alpha : C \longrightarrow J$$

$$P \mapsto P - P_0$$

Remark: We can use any divisor D_0 with $deg(D_0) = 1$ and get a map

$$\alpha : C \longrightarrow J$$

$$P \mapsto P - D_0$$

So $C(K) \subseteq J(K)$. Mordell-Weil implies that $J(K)$ is a finitely generated abelian group when K is a global field. Does it help to understand $C(K)$?

•Today:

Theorem*(Chabauty '38) If K is a number field and $rank_{\mathbf{Z}} J(K) < g$, then $C(K)$ is finite.*

(We use p-adic analysis to prove it)

We can embed K in $\mathbf{Q}_p$ for some prime p. Let's study $J(\mathbf{Q}_p)$ first.

Lemma *There exists a neighborhood of $0 \in J(\mathbf{Q}_p)$ which is isomorphic (as a p-adic analytic group) to $\mathbf{Z}_p^g$.*

More precisely, there are local coordinates $t_1, ..., t_g$ near 0 and power series $\lambda_1, ..., \lambda_g$ in $t_1, ..., t_g$ converging in some neighborhood U of 0 such that $P, Q \in U, P = (t_1(P), ..., t_g(P))$, etc., then $\lambda_i(t_1(P+Q), ..., t_g(P+Q)) = \lambda_i(t_1(P), ..., t_g(P)) + \lambda_i(t_1(Q), ..., t_g(Q))$.

Assume the Lemma for a while.

Proof of Theorem: Let $P_1, ..., P_r$ be generators of the free part of $J(K), r < g$. Replace P_i, if necessary, by $p^m P_i$ so that without loss of generality $P_i \in U$. Consider the vectors

$$\begin{array}{ccc} (\lambda_1(P_1), & \cdots & , \lambda_g(P_1)) \\ \vdots & & \vdots \\ (\lambda_1(P_r), & \cdots & , \lambda_g(P_r)) \end{array}$$

There exist $a_1, ...a_g \in \mathbf{Z}_p$ not all zero such that

$$\sum a_i \lambda_i(P_j) = 0, j = 1, ..., r$$

Let $\lambda = \sum a_i \lambda_i$ which is an analytic function on U. By construction $\lambda(P_i) = 0, i = 1, .., r$ and by the lemma λ is linear in a neighborhood of 0. So $\lambda(P) = 0$ for any $\mathbf{Z}$-linear combination of the $P_i's$.

If $Q \in J(K)$ then $p^m Q \in U$ for some large m. $\lambda(Q) = \frac{1}{p^m}\lambda(p^m Q) = 0$. Suppose by contradiction that $C(K)$ is infinite. $C(K) \subseteq C(\mathbf{Q}_p)$ is compact (C is projective). Hence there is an accumulation point P_0. Take $P_1 \in C(K)$ such that $P_1 - P_0 \in U$. If $P \in C(K)$ is close enough to P_0 then $P - P_0, P_1 - P_0 \in U$ gives $P_1 - P_0 \in J(K) \cap U$. Thus $\lambda(P - P_1) = 0$. But λ is a an analytic function and C is 1-dimensional. So λ can only have finitely many zeros in C, unless the function $\psi : P \mapsto \lambda(P - P_1)$ is identically zero.

If $\psi \not\equiv 0$ we get only finitely many $P's$, contradicting the fact that P_0 was an accumulation point of $C(K)$.

If $\psi \equiv 0$, then $\lambda(P - P_1) = 0, \forall P \in C(K), P - P_1 \in U$. If $Q_1, ..., Q_g \in C(\mathbf{Q}_p)$ are near P_1,

$$\sum \lambda(Q_i - P_1) = 0$$

Now $Q_1 + ... + Q_g - gP_1$ cover an open set of $J(\mathbf{Q}_p)$ as $Q_1, ..., Q_g$ vary. So $\lambda \equiv 0$, a contradiction.□

Idea of Proof of Lemma: Given $P \in J$, translation by P ($Q \mapsto Q + P$) gives a map $\tau_P : J \longrightarrow J$ such that $0 \mapsto P$. The derivative

$$d\tau_P : T_0 J \longrightarrow T_P J$$

is an isomorphism. Therefore the dual spaces $(T_0 J)^*$ and $(T_P J)^*$ are also isomorphic.

Given an element of $(T_0 J)^*$ say v we get a 1-form on J, given by $\omega = (d\tau_P)^*(v)$ i.e., for each P an element of $(T_P J)^*$. We have $\tau_P^* \omega = \omega$. The function

$$\lambda : P \mapsto \int_0^P \omega$$

is linear in P. In fact,

$$\int_0^{P+Q} \omega = \int_0^P \omega + \int_P^{P+Q} \omega = \int_0^P \omega + \int_0^Q \tau_P^* \omega = \int_0^P \omega + \int_0^Q \omega$$

Since $\dim(T_0 J) = g$, we get a g-dimensional set of $\lambda's$.□

Example: $C : y^2 = f(x), deg(f) = 5, g = 2$

Holomorphic differentials on C are generated by $\frac{dx}{y}, \frac{xdx}{y}$. If $t = \frac{x^2}{y}$ we can represent x, y as Laurent series in t.

$$\int_{P_\infty}^P \frac{dx}{y}$$

is a power series in t. Take $\alpha : C \longrightarrow J$. If ω is a differential on J, then $\alpha^* \omega$ is a differential on C.

$$\int_{P_\infty}^{P} \alpha^*\omega = \int_{\alpha(P_\infty)}^{\alpha(P)} \omega$$

(integrals on J transfer to integrals on C)