

Conjecture. Let K be a global field and A/K an Abelian variety. If $X \subset A$ is a closed algebraic set defined over K , then

$$\prod_{v \in M_K} X(K_v) \cap \overline{A(K)} = \overline{X(K)}$$

Here M_K is the set of places of K and \overline{S} represents the closure of the set S in the product topology of $\prod_{v \in M_K} A(K_v)$ where each $A(K_v)$ is endowed with the v -adic metric topology.

Remark. In other words, the conjecture says that if we are given a sequence $P_n \in A(K)$, $n = 1, 2, \dots$, such that for all $v \in M_K$, there exists $Q_v \in X(K_v)$ with $P_n \rightarrow Q_v$ in the v -adic topology then there exists a sequence $\{R_n\} \subset X(K)$ with $R_n \rightarrow Q_v$ in the v -adic topology.

Also notice that if $X(K)$ is finite then $\overline{X(K)} = X(K)$, so we get a stronger conclusion, namely: there exists a $R \in X(K)$ such that $Q_v = R$, for all $v \in M_K$.

We'll sketch the proof of this conjecture in characteristic p under the following assumptions:

- X is a curve with $\text{Genus}(X) \geq 2$, which is not defined over K^p , A is an ordinary abelian variety and the set $\{P \in A(K^{sep}) \mid \exists n \geq 1, p^n P = 0\}$ is finite.

Recall that in this case the Mordell conjecture was proved by writing

$$A(K) = \cup_{i=1}^n (pA(K) + P_i)$$

where P_i , $i = 1, 2, \dots, n$, are coset representatives of $A(K)/pA(K)$. Therefore

$$X \cap A(K) = \cup_{i=1}^n (X \cap (pA(K) + P_i))$$

So we are left to show that each intersection $X \cap (pA(K) + P_i)$ is finite. This is done by first noticing that $pA(K) = V(F(A(K))) = V(A^{(p)}(K^p))$, where V is the dual of the Frobenius map. Let's define a map Φ by $\Phi(P) = V(P) + P_i$ and assume, by contradiction, that $X \cap (pA(K) + P_i)$ is infinite. So $\Phi^{-1}(X) \cap A^{(p)}(K^p)$ is infinite. This would imply, after some work, that $\Phi^{-1}(X)$ is defined over K^p . And so X is defined over K^p , contradicting our previous assumption.

Also notice that the same proof would work if we replace $A(K)$ by its closure in $A(K_v)$ and work with K_v^p instead of K_v , since $K_v^p \cap K = K^p$. Taking another careful look at this proof, one sees that we're really proving that:

- (*) There exists a finite set $Z \subset X$ such that $P_n \in A(K)$ and $P_n \rightarrow P_v \in X(K_v)$ in $A(K_v)$ then $\lim P_n \in Z$.

Namely, there exists a set Z_1 such that $X \cap pA(L) \subset Z$ for any L/K separable extension. Z corresponds to the finitely many cosets of $\overline{A(K)}$ in $p\overline{A(K)}$.

This condition implies that

$$\prod_{v \in M_K} X(K_v) \cap \overline{A(K)} = \prod_{v \in M_K} Z(K_v) \cap \overline{A(K)}$$

which reduces the proof of the conjecture in dimension 1 to proving it in dimension zero. The case of dimension zero requires an extra argument. The condition $\{p \in A(K^{sep}) \mid \exists n \geq 1, p^n P = 0\}$ is finite comes in the proof of the zero dimensional case.

Let's now take a look at the zero characteristic case and assume the condition $\text{rank}(A(K)) < \dim A$ of Chabauty's theorem. In this case we construct $\lambda : A(K_v) \rightarrow K_v$ analytic with $A(K) \subset \lambda^{-1}(0)$, but $X \cap \lambda^{-1}(0)$ is finite. $Z_v = X \cap \lambda^{-1}(0)$ has the property (*). Unfortunately Z_v depends on v in characteristic zero, so we cannot reduce the 1-dimensional case to the 0-dimensional one.

Suppose instead that $\text{rank}(A(K)) < \dim A - 1$, then you can construct $\lambda_1, \lambda_2 : A(K_v) \rightarrow K_v$ linearly independent, with $A(K) \subset \lambda_i^{-1}(0)$. Define $Z_v = X \cap \lambda_1^{-1} \cap \lambda_2^{-1}$.

Conjecture (Stoll). *In this situation, there is a finite set Z independent of v with $Z_v \subset Z$, for all v*

If this is true then the reduction of the 1-dimensional case to the 0-dimensional case works in zero characteristic.