M 328 K 56460 Midterm

1. Describe the set of all solutions x, y in integers to the equation $8x + 15y = 100$ and find all solutions with both x, y positive.

Extended Euclidean algorithm:

$$15 = 8 + 7, 8 = 7 + 1, 7 = 7 \times 1 + 0$$

$$1 = 8 - 7 = 8 - (15 - 8) = 2 \times 8 + (-1) \times 15$$

So $x_0 = 200, y_0 = -100$ is a solution of $8x + 15y = 100$ and the general solution is $x = 200 + 15u, y = -100 - 8u$. To have $x, y > 0$ we must have $u = -13$ which gives $x = 5, y = 4$.

2. Let a be an integer. Show that $gcd(4a, 2a + 3)$ is either 1 or 3. Give examples showing both cases can happen.

Since $2(2a + 3) - 4a = 6$, any common factor of $4a$ and $2a + 3$ must divide 6 so it's one of $1, 2, 3, 6$. Now, $2a + 3$ is odd, so it doesn't have even factors and that rules out $2, 6$, leaving $1, 3$ as the only options.

When $a = 1, gcd(4, 5) = 1$ and when $a = 3, gcd(12, 9) = 3$ so both cases can happen.

3. Let a, b, c be positive integers with $gcd(a, b) = 1$ and $ab = c^3$. Show that there exists integers e, f with $a = e^3, b = f^3$. Give an example where the conclusion doesn't hold if the hypothesis that $gcd(a, b) = 1$ is removed.

We can write $a = \prod p_i^{\alpha_i}, b = \prod q_i^{\beta_i}$ where the p_i, q_i are distinct primes and the α_i, β_i are positive integers. That we can assume that no p_i is a q_j is because $gcd(a, b) = 1$. Then $c^3 = ab = \prod p_i^{\alpha_i} \prod q_i^{\beta_i}$ is the prime factorization of c^3. If $c = \prod r_i^{\gamma_i}$, with distinct primes r_i and γ_i positive integers, then $c^3 = \prod r_i^{3\gamma_i}$ is also the prime factorization of c^3. They have to match by unique factorization so each α_i, β_i is equal to some $3\gamma_j$, so the α_i, β_i are all divisible by 3 and $e = \prod p_i^{\alpha_i/3}, f = \prod q_i^{\beta_i/3}$ are integers with $e^3 = a, f^3 = b$.