M 373 K 57445 First Midterm

name:
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1. Find the decomposition into disjoint cycles and the order of the following element of S_7:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 6 & 1 & 4 & 3 \end{pmatrix}$$

The cycle decomposition is $(125)(37)(46)$ and the order is the least common multiple of the lengths of the cycles, hence it is 6.

2. Let G be a group of order 21. Prove that, for any given $g \in G$, there exists $h \in G, h^2 = g$.

Since G has order 21, every element of G has order dividing 21 by Lagrange's theorem, so every element g of G satisfies $g^{21} = e$. It follows that $g^{22} = g$ for every g in G. We can then take, given g in G, $h = g^{11}$ and $h^2 = (g^{11})^2 = g^{22} = g$.

3. Let G and K be groups. Let $G \times K = \{(g, k) \mid g \in G, k \in K\}$ with operation $(g, k)(g', k') = (gg', kk')$. Let $H = \{(g, e) \mid g \in G\}$, where e is the identity of K. Prove that H is a normal subgroup of $G \times K$, that H is isomorphic to G and that $(G \times K)/H$ is isomorphic to K.

Consider the map $f : G \times K \to K, f((g, k)) = k$. Then

$$f((g, k)(g', k')) = f((gg', kk')) = kk' = f((g, k))f((g', k'))$$

so f is a homomorphism. The kernel of f consists of the (g, k) with $k = f((g, k)) = e$ so the kernel of f is H. Hence H is a normal subgroup of $G \times K$. Also, f is surjective because for example, for any k in K we have (e, k) in $G \times K$ and $f((e, k)) = k$. It now follows from the isomorphism theorem that $(G \times K)/H$ is isomorphic to K.

To show that H is isomorphic to G consider the map $u : G \to H$ defined by $u(g) = (g, e)$, then u is bijective, with inverse $(g, e) \mapsto g$ and

$$u(gg') = (gg', e) = (g, e)(g', e) = u(g)u(g')$$

so u is a homomorphism, so is an isomorphism.