1. Find the decomposition into disjoint cycles and the order of the following element of $S_7$:
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 5 & 7 & 6 & 1 & 4 & 3
\end{pmatrix}
\]

The cycle decomposition is $(125)(37)(46)$ and the order is the least common multiple of the lengths of the cycles, hence it is 6.
2. Let $G$ be a group of order 21. Prove that, for any given $g \in G$, there exists $h \in G, h^2 = g$.

Since $G$ has order 21, every element of $G$ has order dividing 21 by Lagrange’s theorem, so every element $g$ of $G$ satisfies $g^{21} = e$. It follows that $g^{22} = g$ for every $g$ in $G$. We can then take, given $g$ in $G$, $h = g^{11}$ and $h^2 = (g^{11})^2 = g^{22} = g$. 
3. Let $G$ and $K$ be groups. Let $G \times K = \{(g, k) \mid g \in G, k \in K\}$ with operation $(g, k)(g', k') = (gg', kk')$. Let $H = \{(g, e) \mid g \in G\}$, where $e$ is the identity of $K$. Prove that $H$ is a normal subgroup of $G \times K$, that $H$ is isomorphic to $G$ and that $(G \times K)/H$ is isomorphic to $K$.

Consider the map $f : G \times K \rightarrow K, f((g, k)) = k$. Then

$$f(((g, k)(g', k')) = f((gg', kk')) = kk' = f((g, k))f((g', k'))$$

so $f$ is a homomorphism. The kernel of $f$ consists of the $(g, k)$ with $k = f((g, k)) = e$ so the kernel of $f$ is $H$. Hence $H$ is a normal subgroup of $G \times K$. Also, $f$ is surjective because for example, for any $k$ in $K$ we have $(e, k)$ in $G \times K$ and $f((e, k)) = k$. It now follows from the isomorphism theorem that $(G \times K)/H$ is isomorphic to $K$.

To show that $H$ is isomorphic to $G$ consider the map $u : G \rightarrow H$ defined by $u(g) = (g, e)$, then $u$ is bijective, with inverse $(g, e) \mapsto g$ and

$$u(gg') = (gg', e) = (g, e)(g', e) = u(g)u(g')$$

so $u$ is a homomorphism, so is an isomorphism.