M 380 C 59000 Midterm

Name:

1. Let $G =< a, b \mid a^2, b^2, (ab)^4 >$ and $D =< x, y \mid x^4, y^2, xyxy >$. Prove that G is isomorphic to D.

Let F be the free group on generators a, b and define $\phi : F \to D$ by $\phi(a) = y, \phi(b) = xy$ (there are other choices). There is a unique homomorphism $\phi : F \to D$ with these properties. Now

$$\phi(a^2) = y^2 = e, \phi(b^2) = (xy)^2 = e, \phi(((ab)^4) = (yxy)^4 = (x^{-1})^4 = (x^4)^{-1} = e$$

so $\ker \phi$ contains the relations defining G and by the universal property of presentations ϕ induces a homomorphism $\bar{\phi} : G \to D$.

Similarly we can define $\bar{\psi} : D \to G$ by first defining ψ on on the free group on generators x, y by $\psi(x) = ba, \psi(y) = a$ and showing that it induces $\bar{\psi} : D \to G$.

Finally $\bar{\psi}(\bar{\phi}(a)) = \bar{\psi}(y) = a$ and $\bar{\psi}(\bar{\phi}(b) = \bar{\psi}(xy) = baa = b$ so $\bar{\psi} \circ \bar{\phi}$ is the identity on G. Likewise $\bar{\phi} \circ \bar{\psi}$ is the identity on D so they are inverses of each other and G is isomorphic to D.

2. Let G be a finite group, H a normal subgroup of G, p a prime number and S a p-Sylow subgroup of G. Show that $H \cap S$ is a p-Sylow subgroup of H and that HS/H is a p-Sylow subgroup of G/H. Give a counterexample to the first conclusion if H is no longer assumed to be normal in G.

We know that there exists $g \in G$ such that $H \cap gSg^{-1}$ is a p-Sylow subgroup of H. As H is a normal subgroup of G, $H = gHg^{-1}$ so $H \cap gSg^{-1} = g(H \cap S)g^{-1}$ which is isomorphic to $H \cap S$, hence $H \cap S$ has the correct cardinality to be a p-Sylow subgroup of H and since it is a subgroup of H, it has to be a p-Sylow subgroup of H.

By the isomorphism theorem HS/H is isomorphic to $S/H \cap S$, so its cardinality is the correct cardinality to be a p-Sylow subgroup of G/H (since $|G/H| = |G|/|H|$, the highest power of p dividing $|G/H|$ is the quotient of the highest powers of p dividing $|G|$ and $|H|$) since S is a p-Sylow subgroup of G and $H \cap S$ is a p-Sylow subgroup of H. As HS/H is a subgroup of G/H we get the conclusion.

There are many examples. The simplest is $G = S_3, H =< (12) >, S =< (13) >$. S is a 2-Sylow subgroup of G and H is the 2-Sylow subgroup of H but $H \cap S = \{e\}$.

3. Let S be a p-Sylow subgroup of $GL_3(\mathbf{Z}/p)$. Show that S is a solvable non-abelian group of order p^3 and, if $p \neq 2$, that every element of S has order dividing p.

We know that $GL_3(\mathbf{Z}/p)$ has $(p^3-1)(p^3-p)(p^3-p^2)$ elements so the highest power of p dividing this number is p^3 and so a p-Sylow subgroup of $GL_3(\mathbf{Z}/p)$ has order p^3. We also know that p-groups are solvable so a p-Sylow subgroup of $GL_3(\mathbf{Z}/p)$ is solvable.

The group U consisting of the matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with a, b, c running through $\mathbf{Z}/p$ is a subgroup of $GL_3(\mathbf{Z}/p)$ of order p^3. By Sylow's theorems any p-Sylow subgroup of $GL_3(\mathbf{Z}/p)$ is conjugate, in particular isomorphic, to U. So, to finish the proof we need to show that U is non-abelian and all its elements have order dividing p.

Let, for example,

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$gh = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, hg = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so $gh \neq hg$ and U is non-abelian. Also, by induction it is easy to check that

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na & nb + n(n-1)ac/2 \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$$

and, when $n = p > 0$, we get $na = nb = nc = n(n-1)ac/2 = 0$ for all $a, b, c \in \mathbf{Z}/p$, so $g^p = 1$ for all $g \in U$. (A fancier proof can be given using that the characteristic polynomial of any element of U is $(x-1)^3$ and applying the Cayley-Hamilton theorem to get $(g-1)^3 = 0, g \in U$ and conclude that $g^p - 1 = (g-1)^p = 0, p > 2$.)