M 380 C 54580 First Midterm

Name:

Do three out of the four questions below and please indicate here which questions you chose:

1. Let p, q be distinct primes and G a group of order p^2q. Prove that G has either a normal p-Sylow subgroup or a normal q-Sylow subgroup.

Let n_p be the number of p-Sylow subgroups of G. Then we know, by Sylow's theorems that $n_p \equiv 1 (\bmod p)$ and $n_p | q$. If $n_p = 1$, we are done, since the unique p-Sylow subgroup is normal, by Sylow's theorems. Otherwise, $n_p = q \equiv 1 (\bmod p)$, which implies $q > p$. We also know, by Sylow's theorems that $n_q \equiv 1 (\bmod q)$ and $n_q | p^2$. Again we are done if $n_q = 1$. We cannot have $n_q = p \equiv 1 (\bmod q)$, since $q > p$. The only other possibility is then $n_q = p^2$. The q-Sylow subgroups have order q prime so they intersect only at the identity (since the intersection is a subgroup) and every non-identity element of such a subgroup has order q, therefore G has $p^2(q-1)$ elements of order q. The other elements of G number $p^2q - p^2(q-1) = p^2$. A p-Sylow subgroup of G has p^2 elements and does not have any element of order q, since q does not divide p^2, so it must consist of these remaining p^2 elements of G and is thus unique, hence normal, by Sylow's theorems.

2. Show in detail that a finite abelian group is solvable, that is, has a composition series whose successive quotients are cyclic of prime order.

By the Jordan-Hölder theorem, any finite group has a composition series whose successive quotients are simple. Such a quotient G/N is abelian since G is a subgroup of our group, so abelian and $NaNb = Nab = Nba = NbNa$ for any $a, b \in N$, that is G/N is abelian. So we need to show that a simple finite abelian group A is cyclic of prime order. Let $x \in A, x \neq 1$, then $< x >$ is a normal subgroup of A, so $< x >= A$. Let n be the order of x. If n is not prime $n = ab, a > 1, b > 1$ and $< x^a >$ is a non-trivial proper subgroup of A, contradiction, so n is prime and we are done.

3. Let G be a finite group acting on a finite set S and assume that the action is transitive, that is, has only one orbit. Let H be a *normal* subgroup of G and consider H acting on S by restriction of the action of G. Prove that all orbits of H have the same cardinality. Prove also that, for $s \in S$, the number of orbits for the action of H is equal to $|G|/|HG_s|$, where G_s is the stabilizer of s in G.

If $s, t \in S$ there exists g in G with $t = gs$, by hypothesis. If O_s, O_t are the H-orbits of s and t respectively, I claim that $x \mapsto gx$ is a bijection between O_s and O_t. Indeed if $hs \in O_s$ then $ghg^{-1} = h' \in H$, since H is normal in G, so $ghs = h'gs = h't \in O_t$, so $x \mapsto gx$ maps O_s to O_t. By the same argument, $x \mapsto g^{-1}x$ maps O_t to O_s. Clearly, these two maps are inverses of each other and and this establishes the bijection.

S is the unique orbit of G so its cardinality is $|S| = |G|/|G_s|$. By the first part, each H-orbit has cardinality $|H|/|H_s|$, so the number of orbits is $|G||H_s|/|H||G_s|$. Now, $H_s = \{h \in H \mid hs = s\} = H \cap G_s$ and, since H is normal in G we can apply the diamond isomorphism theorem to conclude that HG_s/H is isomorphic to G_s/H_s. In particular, $|HG_s| = |H||G_s|/|H_s|$ and the result now follows from the previous expression for the number of orbits.

For an example that shows that the hypothesis that H is normal in G is necessary, take $G = S_3$ acting on $\{1, 2, 3\}$ as usual, and $H = \{1, (23)\}$, the stabilizer of 1. Then the orbits of H are $\{1\}, \{2, 3\}$ and they don't have the same cardinality.

4. Show that $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$ has $p+1$ subgroups of order p, when p is prime. Show that the group of automorphisms of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ is isomorphic to S_3 by considering its action on the subgroups of order 2.

Every element of $G = \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$, except the identity, has order p, so it generates a subgroup of order p of G. Two distinct such subgroup meet only in the identity since p is prime (since the intersection is a subgroup). Thus the non-zero elements of G are partitioned into the sets of non-zero elements of the subgroups of order p and each such set has $p-1$ elements. As G has p^2 elements we conclude that there are $(p^2-1)/(p-1) = p+1$ such subgroups.

By the above $G = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ has three subgroups of order 2. If ϕ is an automorphism of G and H is a subgroup of order 2, then so is $\phi(H)$, since ϕ is injective and is a homomorphism. Thus ϕ induces a permutation of these subgroups of order 2. We thus get a homomorphism $\lambda : Aut(G) \to S_3$. If ϕ acts trivially on the subgroups, then $\phi(\{e, x\}) = \{e, x\}, x \in G, x \neq e$, so $\phi(x) = x$ for all such x since $\phi(e) = e$ always, thus ϕ is the identity. Therefore λ is injective. Now $Aut(G)$ has 6 elements, as can be seen directly, e.g., by noticing that $Aut(G) = GL_2(\mathbf{Z}/2\mathbf{Z})$, and S_3 also has 6 elements so λ is a bijection and thus is an isomorphism.