M 325 K 55165 First Midterm

name:

Write on the space alloted for each question and continue on the back of the page if necessary.

1. Write a truth table to show that $\sim (p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

As the collumns corresponding to $\sim (p \vee q)$ and $\sim p \wedge \sim q$ are identical, the two are logically equivalent.

2. A sequence a_n of real numbers has limit $+\infty$ if the following statement is true:
$\forall M > 0, \exists n_0, \forall n > n_0, a_n > M$.

Write a statement (without using the symbol $\sim$) that says that the sequence a_n does **not** have limit $+\infty$.

$$\exists M > 0, \forall n_0, \exists n > n_0, a_n \leq M$$

3. Prove that if A, B, C are sets, then $C - (A \cup B) = (C - A) \cap (C - B)$.

Two solutions. First using logic. $x \in C - (A \cup B)$ means that $x \in C \wedge x \notin (A \cup B)$. Now $x \notin (A \cup B)$ is the same as $\sim (x \in A \vee x \in B)$, which is the same as $x \notin A \wedge x \notin B$ by de Morgan's law (problem 1). Putting all these together we get that $x \in C - (A \cup B)$ is the same as $x \in C \wedge x \notin A \wedge x \notin B$ which is now the same as $x \in C \wedge x \notin A \wedge x \in C \wedge x \notin B$, that is $x \in C - A \wedge x \in C - B$ which is $x \in (C - A) \cap (C - B)$, as was to be shown.

The second solution uses set equations. $C - (A \cup B) = C \cap (A \cup B)^c = C \cap (A^c \cap B^c)$ by de Morgan's law for sets. Now $C \cap (A^c \cap B^c) = C \cap A^c \cap C \cap B^c$ and the latter is $(C - A) \cap (C - B)$.

4. Prove by induction that $\sum_{i=1}^{n} i^2 \leq n^3$ for all $n \geq 1$.

Base case $n = 1$: $\sum_{i=1}^{1} i^2 = 1^2 \leq 1^3$.

Induction step: If $\sum_{i=1}^{n} i^2 \leq n^3$ then $\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 \leq n^3 + (n+1)^2$ by the induction hypothesis.

So if we show that $n^3 + (n+1)^2 \leq (n+1)^3$, then we can conclude that $\sum_{i=1}^{n+1} i^2 \leq (n+1)^3$. Well, $(n+1)^3 = n^3 + 3n^2 + 3n + 1$ and $n^3 + (n+1)^2 = n^3 + n^2 + 2n + 1$. The difference between the former and the latter is $n^3+3n^2+3n+1-(n^3+n^2+2n+1) = 2n^2+n$ which is positive for $n \geq 1$, so indeed $n^3 + (n+1)^2 \leq (n+1)^3$.