M 373 K 57445 Second Midterm

name:
signature:

1. Let R be a commutative ring and a, b elements of R. Prove that $I = \{xa + yb \mid x, y \in R\}$ is an ideal of R. Assume now that I as above satisfies $I = \{xd \mid x \in R\}$ for some $d \in R$. Prove that both a, b are multiples of d and also that, if $c \in R$ is such that both a, b are multiples of c, then d is a multiple of c.

First we need to prove that I is a subgroup of the additive group of R. Now, $0 = 0a+0b$ so $0 \in I$. If $z = xa + yb \in I$ then $-z = (-x)a + (-y)b$ is also in I. If $z = xa + yb, z' = x'a + y'b \in I$ then $z + z' = xa + yb + x'a + y'b = (x + x')a + (y + y')b$ is in I and I is a subgroup of R. To show that I is an ideal of R, it remains to check that if $z = xa+yb \in I$ and $w \in R$, then $zw \in I$, but $zw = (xa + yb)w = (xw)a + (yw)b$ is in I so we are done with showing that I is an ideal of R.

Suppose now that $I = \{xd \mid x \in R\}$ for some $d \in R$. So I consists of the multiples of d. To show that both a, b are multiples of d we need to show that $a, b \in I$. We have $a = 1a + 0b, b = 0a + 1b$ so they are indeed in I.

If I consists of the multiples of d, then $d = 1d$ is in I, so $d = ax + by$ for some $x, y \in R$. If both a, b are multiples of c, then $a = cu, b = cv$ for some $u, v \in R$ and so $d = cux + cvy = c(ux + vy)$ is a multiple of c, as was to be shown.

2. Let R be a commutative ring. We say that an ideal I of R is minimal if the only ideals of R contained in I are $\{0\}$ and I itself. Prove that a minimal ideal is principal, that is, there exists $a \in R$ such that $I = \{xa \mid x \in R\}$.

If $I = \{0\}$ then $a = 0$ works. Otherwise there exists some $a \neq 0$ in I. We show that this a works. As I is an ideal $xa \in I$ for all $x \in R$, so $\{xa \mid x \in R\} \subset I$. As $\{xa \mid x \in R\}$ is an ideal of R, it follows from the minimality of I that $\{xa \mid x \in R\} = \{0\}$ or I. As $a = 1a$ is nonzero and an element of $\{xa \mid x \in R\}$, this set cannot be $\{0\}$ so it must be I and we are done.

3. Let $\mathbf{R}$ denote the real numbers and $R = \mathcal{F}(\mathbf{R}, \mathbf{R})$ be the ring of all real functions of a real variable. Show that if $f \in R$, there exists $g \in R, g^3 = f$. Conclude from this that R does not have any irreducible elements.

We know, from Calculus, that every real number has a unique real cube root. So, if $f \in R$ we can define $g \in R$ by $g(x) = f(x)^{1/3}, x \in \mathbf{R}$. Then, for all $x \in \mathbf{R}, g(x)^3 = f(x)$ and therefore $g^3 = f$.

Assume by contradiction that f is an irreducible element of R. By the first part we can find $g \in R, g^3 = f$. So we have $f = gg^2$ and since f is irreducible either g or g^2 is a unit in R. If g^2 is a unit then there exists h in R with $ggh = g^2h = 1$ so gh is an inverse of g and g is a unit in R. Either way, g is a unit so there exists $k \in R, kg = 1$. Then $k^3f = k^3g^3 = (kg)^3 = 1$ and we conclude that f is a unit. But, by definition, irreducible elements are not units, so we arrived at a contradiction which completes the proof.