

M 325 K 55165 Second Midterm

name:

Write on the space allotted for each question and continue on the back of the page if necessary.

1. Let X, Y, Z be sets, $f, g : X \rightarrow Y$ and $h : Y \rightarrow Z$ be functions. Prove that, if h is one-to-one and $h \circ f = h \circ g$, then $f = g$. Also, give an example of f, g, h (with h not one-to-one) such that $f \neq g$ but $h \circ f = h \circ g$.

If h is one-to-one, then for any $y_1, y_2 \in Y$, if $h(y_1) = h(y_2)$, then $y_1 = y_2$. If $h \circ f = h \circ g$ and $x \in X$, then $h(f(x)) = h \circ f(x) = h \circ g(x) = h(g(x))$, therefore $f(x) = g(x)$. Since this is true for all $x \in X$, we get $f = g$.

There are lots of examples. The simplest is with $X = Z = \{1\}, Y = \{1, 2\}$. Take $f(1) = 1, g(1) = 2, h(1) = h(2) = 1$. A “calculus” example is $f(x) = x, g(x) = -x, h(x) = x^2$, with $X = Y = Z$ being the real numbers.

2. Find the coefficient of x^2 in the expansion of $(x - x^{-1})^{10}$.

$$(x - x^{-1})^{10} = \sum_{r=0}^{10} \binom{10}{r} x^r (-x^{-1})^{10-r}$$

Now, $x^r (-x^{-1})^{10-r} = \pm x^{2r-10}$ and the exponent is 2 only when $2r - 10 = 2$, that is, $r = 6$. Also $10 - r = 4$ is even so the sign is $+$. So the coefficient of x^2 is $\binom{10}{6} = 210$.

3. Let X be a set and R a relation on X . Assume that R is reflexive and transitive. For any a in X , define $[a] = \{x \in X \mid aRx\}$. Prove that, for any $a, b \in X$, aRb if and only if $[b] \subseteq [a]$.

Assume aRb and let $x \in [b]$, then bRx . From both aRb and bRx we get aRx because R is transitive. So $x \in [a]$. As $x \in [b]$ was arbitrary, we get $[b] \subseteq [a]$.

Assume $[b] \subseteq [a]$. Since R is reflexive, bRb so $b \in [b]$, therefore $b \in [a]$. This means that aRb .

4. Let $X = \{1, 2, 3, 4, 5\}$. Determine the number of functions $f : X \rightarrow X$ which satisfy $f(1) \neq f(2)$.

There are 5 choices for $f(1)$ and, once $f(1)$ is chosen, only 4 choices for $f(2)$ since $f(1) \neq f(2)$. There are 5 choices for each of $f(3), f(4), f(5)$, so by the multiplication rule, there are a total of $5 \times 4 \times 5 \times 5 \times 5 = 2500$.