1. Let G be a finite abelian group and define

$$G^* = \{f : G \to \mathbf{C}^* | f \text{ homomorphism}\},$$

where $\mathbf{C}^*$ is the group of non-zero complex numbers under multiplication. For $f, g \in G^*$ define $fg \in G^*$ by

$$fg(\gamma) = f(\gamma)g(\gamma), \forall \gamma \in G.$$

(a) Prove that the above operation makes G^* into a group.
(b) Prove that if G is cyclic, then G is isomorphic to G^*.
(c) Prove that if G, H are finite abelian groups then $(G \times H)^*$ is isomorphic to $G^* \times H^*$.
(d) Prove that if G is a product of finite cyclic groups then G is isomorphic to G^*.
(e) Prove that, if G is a finite abelian group and $\gamma \in G$, then the map $\gamma^* : G^* \to \mathbf{C}^*, f \mapsto f(\gamma)$ is an element of $(G^*)^*$. Prove that the map $\gamma \mapsto \gamma^*$ is an isomorphism $G \to (G^*)^*$.