

An R -module is projective if it is a direct summand of a free R -module. An R -module M is torsion-free if for $r \in R, m \in M, rm = 0$ implies $r = 0$ or $m = 0$. An R -module M is free (on a set $X \subset M$) if every element of M can be written uniquely as a finite linear combination of elements of X .

1. Let R be a domain and I and J ideals of R . Prove that I and J are isomorphic as R -modules if and only if there exists $a, b \in R$ such that $aI = bJ$.

2. Prove that \mathbf{Q} is a torsion-free \mathbf{Z} -module but is not a free \mathbf{Z} -module. Prove that \mathbf{Q}/\mathbf{Z} is a torsion \mathbf{Z} -module and that the n -torsion submodule of \mathbf{Q}/\mathbf{Z} is isomorphic to $\mathbf{Z}/n\mathbf{Z}$.