A Few Basic Concepts

Definition 1. *A projective plane is defined as a set of elements,called points, together with distinguished sets of points,called lines, as well as a relation between points and lines subject to the following conditions:*
1. Every pair of distinct lines is incident with a unique point.
2. Every pair of distinct points is incident with a unique line.(i.e,to every pair of distinct points there is exactly one line which contains both points);
3. There exists four points such that no three of them are incident with a single line.

Remark. If Π is a finite projective plane then:
1. There exist $m \geq 2$ such that every point (line) is incident with exactly $m+1$ lines (points) of Π.
2. Π consists of exactly m^2+m+1 points (lines).
m is called the order of the finite projective plane. For every prime power $q = p^n$, p prime, there exists a finite projective plane of order q.

Definition 2. *An oval in a projective plane of order q is a set of points which meets every line in at most two points, and has a unique tangent (a line meeting it in one point only) at each of its points.The number points is q+1. (For, if P is a point of the oval, then the q non-tangent lines through P each contains one further point of the oval.)*

Definition 3. *In a projective plane , a conic is the set of zeros of a non-singular quadratic form.*

Any conic is an oval. If q is odd then the converse statement is a celebrated theorem of Segre.

The Lemma

Lemma 1. *Every inscribed triangle and its mate are in perspectivity*

Proof. We may take the inscribed triangle as the reference for a homogenous coordinisation of Π. Thus the vertices of the triangles are:

$$A_1 =< (1,0,0) >, A_2 =< (0,1,0) >, A_3 =< (0,0,1) > .$$

Let a_1, a_2, a_3 be tangents to the oval at A_1, A_2, A_3 respectively.Hence, the equations of the tangents are of the form

$$a_1 : x_2 = k_1 x_3,\ a_2 : x_3 = k_2 x_1,\ a_3 : x_1 = k_3 x_2, \tag{1}$$

where k_1,k_2,k_3 are non-zero elements of F (the finite field of order q).

Let $B =< (c_1, c_2, c_3) >$ be any of the $q-2$ points of the oval distinct from A_1,A_2,A_3. Then $c_1c_2c_3 \neq 0$ because the lines of the triangle already intersect the oval at two points; namely, the vertices of the triangle. Now A_1B, A_2B, A_3B have respective equations

$$x_2 = h_1 x_3,\ x_3 = h_2 x_1,\ x_1 = h_3 x_2, \tag{2}$$

where $h_1 = c_2{c_3}^{-1}, h_2 = c_3{c_1}^{-1}, h_3 = c_1{c_2}^{-1}$ and $h_i \neq k_i, i = 1, 2, 3$. They clearly satisfy

$$h_1h_2h_3 = 1. \tag{3}$$

Let h_1 be any non-zero element of $F - \{k_1\}$. Now consider the line $x_2 = h_1x_3$. Since this line is not the tangent at A_1 it meets the oval at A_1 and another point B (say). So for each $h_1 \in F - \{k_1, 0\}$ we get a line through A_1 and some point in the oval . Now for different values of h_1 we get different points of the oval because any line intersects the oval at atmost two points. So there exists an injection between non-tangent lines through A_1 and the points of the oval other than A_1 .Since both the sets have cardinality q, we have a bijection,

$$f_1 : L_{A_1} - \{a_1\} \to O - \{A_1\}.$$

Similarily we have bijections

$$f_2 : L_{A_2} - \{a_2\} \to O - \{A_2\},$$

$$f_3 : L_{A_3} - \{a_3\} \to O - \{A_3\}.$$

So composing these we get bijections,

$$h_2 : L_{A_1} - \{a_1,\ A_1A_2,\ A_1A_3\} \to L_{A_2} - \{a_2,\ A_2A_1,\ A_2A_3\},$$

$$h_3 : L_{A_1} - \{a_1,\ A_1A_2,\ A_1A_3\} \to L_{A_3} - \{a_3,\ A_3A_2,\ A_3A_1\}.$$

So $x_2 = h_1x_3$ uniquely determines the lines $x_3 = h_2x_1$, $x_1 = h_3x_2$, and since they all meet at the point B we have

$$h_1h_2h_3 = 1. \tag{4}$$

Vary h_1 over $F - \{k_1\}$ to get unique h_2,h_3 for each value. Further, they satisfy the relationship above. Now multiply these $q - 2$ relations to obtain

$$\prod_{h_1 \in F - \{k_1, 0\}} h_1 h_2(h_1) h_3(h_1) = 1. \tag{5}$$

Now, finally we shall use the fact that q is odd. We know that all the non-zero elements of F satisfy $x^{q-1} - 1 = 0$. Because q is odd the product of the roots is -1.Let P denote the product of the roots. Then $P^3 = -1$. But relation (5) yields that

$$k_1k_2k_3 = -1 \tag{6}$$

Using equation (6) and the equations of the lines A_1P,A_2Q and A_3R (refer to the fig 2.The point B should read K) we can conclude that they intersect at $K =< (1, k_1k_2, -k_2) >$ and the lemma is proved.

□

A Few Calculuations

Without loss of generality assume that K is the point $< (1,1,1) >$ which implies that

$$k_1 = k_2 = k_3 = -1.$$

If $B =< (c_1, c_2, c_3) >$ is any of the q-2 points of the Oval distinct from A_1,A_2,A_3.Let b denote, $b_1x_1 + b_2x_2 + b_3x_3 = 0$ the tangent at B. Since B lies on b and A_1,A_2,A_3 do not lie on b we have

(7) $$b_1c_1 + b_2c_2 + b_3c_3 = 0 \text{ and } b_i \neq 0 \; for \; i = 1, 2, 3.$$

The direction ratio for PB: $(-c_2 + c_3, c_1 + c_3, -c_1 - c_2)$
The equation of this line is: $(-c_2 + c_3)x_1 + (c_1 + c_3)x_2 + (-c_1 - c_2)x_3 = 0$
The direction ratio for RA_3: $(b_3 - b_1, -b_2, 0)$
The equation of this line is: $(b_3 - b_1)x_1 - b_2x_2 = 0$
The direction ratio for QA_2: $(b_1 - b_2, 0, b_3)$
The equation of this line is: $(b_1 - b_2)x_1 + b_3x_3 = 0$
(refer to the fig 1. Disregard the point K)
Because $\triangle\ BA_2A_3$ and $\triangle\ PQR$ are in perspective (by the lemma)and since we are looking for a non-trivial solution the determinant below must equal 0.

$$\begin{vmatrix} -c_2 + c_3 & c_1 + c_3 & -c_1 - c_2 \\ b_3 - b_1 & -b_2 & 0 \\ b_1 - b_2 & 0 & b_3 \end{vmatrix} = 0,$$

$$\Rightarrow (b_1 - b_2 - b_3)((c_1 + c_3)b_3 - (c_1 + c_2)b_2) = 0.$$

But since P does not lie on b we have, $b_1 - b_2 - b_3 \neq 0$

(8) $$\Rightarrow (c_1 + c_3)b_3 = (c_1 + c_2)b_2.$$

A similar consideration for $\triangle\ BA_3A_1$,$\triangle\ BA_2A_1$ and their mates we have

(9) $$\Rightarrow (c_2 + c_3)b_3 = (c_1 + c_2)b_1 \text{ and } (c_1 + c_3)b_1 = (c_3 + c_2)b_2.$$

But these equations imply that

(10) $$b_1 : b_2 : b_3 = (c_2 + c_3) : (c_1 + c_3) : (c_1 + c_2).$$

Substituting in equation (7) we have

(11) $$2(c_1c_2 + c_2c_3 + c_3c_2) = 0,$$

which since q is odd, gives

(12) $$c_1c_2 + c_2c_3 + c_3c_2 = 0$$

Since each of the $q-2$ points of the oval together with A_1,A_2,A_3 satisfy the equation of the conic

$$x_1x_2 + x_2x_3 + x_3x_2 = 0$$

and since the conic has exactly $q + 1$ points we are done.