1. Let R be a domain and I and J ideals of R. Prove that I and J are isomorphic as R-modules if and only if there exists $a, b \in R$ such that $aI = bJ$.

First, a correction to the statement. We need to assume a, b are non-zero.

Now, if $aI = bJ$, given $x \in I$ we have $ax \in aI = bJ$ so there exists $y \in J, ax = by$. Since R is a domain, y is uniquely determined by x and so we can let $\phi(x) = y$ and this defines $\phi : I \to J$. It is easy to show that $\phi(x)$ is an R-module homomorphism. E.g.

$$b\phi(x_1 + x_2) = a(x_1 + x_2) = ax_1 + ax_2 = b\phi(x_1) + b\phi(x_2)$$

and since R is a domain, $\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$ and so on. If $ax = b0 = 0$ then $x = 0$ so ϕ is injective and the same construction with a, b reversed gives an inverse for ϕ so it is also surjective.

Now suppose there exists a R-module isomorphism $\phi : I \to J$. As the result is obvious if $I = (0)$, we may assume there exists $b \neq 0, b \in I$ and we let $a = \phi(b) \in J, a \neq 0$. Now let's prove that $aI = bJ$. Given $x \in I$ we have $ax = \phi(b)x = x\phi(b) = \phi(bx) = b\phi(x)$ using that $\phi(x)$ is an R-module homomorphism and that $x, b \in I \subset R$. It follows that $aI \subset bJ$, since $\phi(x) \in J$. Using the inverse of $\phi(x)$ in the same manner, we get $bJ \subset aI$ and we are done.