

Galois groups of quartics over $\mathbf{C}(z)$

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This is an explanatory note to go with the website:

<http://www.ma.utexas.edu/users/voloch/galois.html>

Let $P(z, w)$ be an irreducible polynomial in two variables and complex coefficients. We will view it as a polynomial in one variable w and coefficients in $\mathbf{C}(z)$. Let n be its degree, which we assume positive, K its splitting field and G its Galois group.

For a given complex number z_1 , the polynomial $P(z_1, w)$ has n distinct roots in \mathbf{C} if and only if its discriminant doesn't vanish. It is clear that the discriminant is a polynomial in z_1 and we call the set of zeros of this polynomial the branch locus of P , denoted B and its elements the branch points. (We should worry about infinity also, but won't for now).

Fix a complex number $z_0 \notin B$. The polynomial $P(z_0, w)$ has n distinct roots in \mathbf{C} and for each of these roots, the implicit function theorem allows us to define an analytic function $w_i(z)$ in a neighbourhood of z_0 such that $w_i(z), i = 1, \dots, n$ are the roots of $P(z, w) = 0$. Thus K is embedded in the field of meromorphic functions of this neighbourhood of z_0 .

Let now γ be a loop on $\mathbf{C} \setminus B$ starting and ending at z_0 . We can analytically continue each $w_i(z)$ along γ and since the equation $P(z, w_i(z)) = 0$ will analytically continue as zero, when we return to z_0 we have effected a permutation of the $w_i(z)$ but we have done more than that, since following the continuation of an element of K gives a map $\rho(\gamma) : K \rightarrow K$, which is easily seen to be an automorphism of $K/\mathbf{C}(z)$. It is also clear that $\rho(\gamma)$ depends only on the homotopy class of γ . Thus we have defined a map $\rho : \pi_1(\mathbf{C} \setminus B, z_0) \rightarrow G$. It is easy to see that ρ is a homomorphism and we call it the monodromy representation.

Theorem. *The monodromy representation is surjective.*

Proof: Let H be image of ρ and f a function in the fixed field of H inside K . By definition of ρ , this means that analytic continuation of f along any loop on $\mathbf{C} \setminus B$ gives f back again. It follows (see e.g. [A] Ch. 8, 1.6) that f can be extended to a meromorphic function on $\mathbf{C} \cup \{\infty\}$ and so f is a rational function of z (see e.g. [A] Ch. 4, 3.2, exercise

2). That is, the fixed field of H inside K is $\mathbf{C}(z)$ and, by Galois theory, we conclude that $H = G$.

To compute $\rho(\gamma)$ we subdivide γ in small steps and at each step we compute the values of $w_i(z)$ at that step using the previous values as initial approximations in Newton's method, which iteratively replaces an approximation w to a root by $w - P(z, w)/(\partial P/\partial w)(z, w)$. ■
 In the program we use eight iterations and that seems ample precision as long as the steps are suitably small.

Theorem. *Every finite group is a Galois group of an extension of $\mathbf{C}(z)$.*

Proof: If G is a finite group, then is finitely generated. The fundamental group of $\mathbf{C} \setminus B$ is the free group on $\#B$ letters, for any finite set B . So, for sufficiently large B , we can write G as a quotient of $\pi_1(\mathbf{C} \setminus B)$ by a subgroup Γ . Then Γ acts on the universal cover U of $\mathbf{C} \setminus B$ and U/Γ is a cover of $\mathbf{C} \setminus B$ with group G . A (hard) theorem of Riemann then proves that U/Γ is the Riemann surface of an algebraic function (see e.g. [A] Ch. 8, 2.3, Theorem 4).

For our examples, we didn't need to use this theorem, however. The transitive subgroups of S_4 are S_4, A_4, D_4, K_4, C_4 respectively the symmetric, alternating, dihedral, Klein and cyclic groups. These groups all act on $\mathbf{C} \cup \{\infty\}$ by Möbius transformations and we produced our polynomials this way. In all cases we can write z as a rational function of w . The polynomials are: $P(z, w) = w^4 - zw - 2$ (symmetric), $w^4 - zw^2 - 1$ (klein), $w^4 - z - 2$ (cyclic), $w^4 - 2w^2 - z - 3$ (dihedral) and $81w^4 - 216w^2 + (192z - 384)w + (192z - 240)$ (alternating).

Some of these ideas are also discussed in [Ku]. The reader may also want to look at Klein's famous monograph [K] where he discusses a similar action of A_5 on $\mathbf{C} \cup \{\infty\}$ and the collection of articles [L] which elaborate on Klein. Another interesting reference, perhaps only tangentially relevant, is [DM].

References.

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