

Visible Points on Curves over Finite Fields

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April 18, 2007

Abstract

For a prime p and an absolutely irreducible modulo p polynomial $f(U, V) \in \mathbb{Z}[U, V]$ we obtain an asymptotic formulas for the number of solutions to the congruence $f(x, y) \equiv a \pmod{p}$ in positive integers $x \leq X$, $y \leq Y$, with the additional condition $\gcd(x, y) = 1$. Such solutions have a natural interpretation as solutions which are visible from the origin. These formulas are derived on average over a for a fixed prime p , and also on average over p for a fixed integer a .

1 Introduction

Let p be a prime and let $f(U, V) \in \mathbb{Z}[U, V]$ be a bivariate polynomial with integer coefficients.

For real X and Y with $1 \leq X, Y \leq p$ and an integer a we consider the set

$$\mathcal{F}_{p,a}(X, Y) = \{(x, y) \in [1, X] \times [1, Y] : f(x, y) \equiv a \pmod{p}\}$$

which the set of points on level curves of $f(U, V)$ modulo p .

If the polynomial $f(x, y) - a$ is nonconstant absolutely irreducible polynomial modulo p of degree bigger than one can easily derive from the Bombieri bound [1] that

$$\#\mathcal{F}_{p,a}(X, Y) = \frac{XY}{p} + O(p^{1/2}(\log p)^2), \quad (1)$$

where the implied constant depends only on $\deg f$, see, for example, [3, 4, 9, 11].

In this paper we consider an apparently new question of studying the set

$$N_{p,a}(X, Y) = \{(x, y) \in \mathcal{F}_{p,a}(X, Y) : \gcd(x, y) = 1\}.$$

These points have a natural geometric interpretation as points on $\mathcal{F}_{p,a}(X, Y)$ which are “visible” from the origin, see [2, 6, 7, 10] and references therein for several other aspects of distribution of visible points in various regions.

We show that on average over $a = 0, \dots, p-1$, the cardinality $N_{p,a}(X, Y)$ is close to its expected value $6XY/\pi^2 p$, whenever

$$XY \geq p^{3/2+\varepsilon} \quad (2)$$

for any fixed $\varepsilon > 0$ and sufficiently large p .

We then consider the dual situation, when a is fixed (in particular we take $a = 0$) but p varies through all primes up to T .

We recall $A \ll B$ and $A = O(B)$ both mean that $|A| \leq cB$ holds with some constant $c > 0$, which may depend on some specified set of parameters.

2 Absolute Irreducibility of Level Curves

We start with the following statement which could be of independent interest.

Lemma 1. *If $F(U, V) \in \mathbb{K}[U, V]$ is absolutely irreducible of degree n over a field \mathbb{K} , then $F(U, V) - a$ is absolutely irreducible for all but at most $C(n)$ elements $a \in \mathbb{K}$, where $C(n)$ depends only on n .*

Proof. The set of polynomials of degree n is parametrized by a projective space $\mathbb{P}^{s(n)}$ of dimension $s(n) = (n+1)(n+2)/2$ over \mathbb{K} , coordinatized by

the coefficients. The subset X of $\mathbb{P}^{k(n)}$ consisting of reducible polynomials is a Zariski closed subset because it is the union of the images of the maps

$$\mathbb{P}^{s(k)} \times \mathbb{P}^{s(n-k)} \rightarrow \mathbb{P}^{s(n)}, \quad k \leq n/2,$$

given by multiplying a polynomial of degree k with a polynomial of degree $n-k$. The map $t \mapsto F(U, V) - t$ describes a line in $\mathbb{P}^{s(n)}$ and by the assumption of absolute irreducibility of F , this line is not contained in X . So, by the Bézout theorem, it meets X in at most $C(n)$ points, where $C(n)$ is the degree of X . Hence for all but at most $C(n)$ values of a , $F(U, V) - a$ is absolutely irreducible. \square

3 Visible Points on Almost All Level Curves

Throughout this section, the implied constants in the notations $A \ll B$ and $A = O(B)$ may depend on the degree $n = \deg f$.

Theorem 2. *Let f be a polynomial with integer coefficients which is absolutely irreducible and of degree bigger than one modulo the prime p . Then for real X and Y with $1 \leq X, Y \leq p$ we have*

$$\sum_{a=0}^{p-1} \left| N_{p,a}(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll X^{1/2} Y^{1/2} p^{3/4} \log p.$$

Proof. Let \mathcal{A}_p consist of $a \in \{0, \dots, p-1\}$ for which $f(U, V) - a$ is absolutely irreducible modulo p .

For an integer d , we define

$$M_{p,a}(d; X, Y) = \#\{(x, y) \in \mathcal{F}_{p,a}(X, Y) \mid \gcd(x, y) \equiv 0 \pmod{d}\}.$$

Let $\mu(d)$ denote the Möbius function. We recall that $\mu(1) = 1$, $\mu(d) = 0$ if $d \geq 2$ is not square-free and $\mu(d) = (-1)^{\omega(d)}$ otherwise, where $\omega(d)$ is the number of distinct prime divisors d . By the inclusion-exclusion principle, we write

$$N_{p,a}(X, Y) = \sum_{d=1}^{\infty} \mu(d) M_{p,a}(d; X, Y). \quad (3)$$

Writing

$$x = ds \quad \text{and} \quad y = dt,$$

we have

$$\#M_{p,a}(d; X, Y) = \#\{(s, t) \in [1, X/d] \times [1, Y/d] \mid f(ds, dt) \equiv a \pmod{p}\}.$$

Thus $M_{p,a}(d; X, Y)$ is the number of points on a curve in a given box. If $a \in \mathcal{A}_p$ and $1 \leq d < p$ then $f(dU, dV) - a$ remains absolutely irreducible modulo p . Accordingly, we have an analogue of (1) which asserts that

$$M_{p,a}(d; X, Y) = \frac{XY}{d^2p} + O(p^{1/2}(\log p)^2). \quad (4)$$

We fix some positive parameter $D < p$ and substitute the bound (4) in (3) for $d \leq D$, getting

$$\begin{aligned} N_{p,a}(X, Y) &= \sum_{d \leq D} \left(\frac{\mu(d)XY}{d^2p} + O(p^{1/2}(\log p)^2) \right) + O\left(\sum_{d > D} M_{p,a}(d; X, Y) \right) \\ &= \frac{XY}{p} \sum_{d \leq D} \frac{\mu(d)}{d^2} + O\left(Dp^{1/2}(\log p)^2 + \sum_{d > D} M_{p,a}(d; X, Y) \right) \end{aligned}$$

for every $a \in \mathcal{A}_p$.

Furthermore

$$\sum_{d \leq D} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(D^{-1}) = \prod_{\ell} \left(1 - \frac{1}{\ell^2} \right) + O(D^{-1}),$$

where the product is taken over all prime numbers ℓ . Recalling that

$$\prod_{\ell} \left(1 - \frac{1}{\ell^2} \right) = \zeta(2)^{-1} = \frac{6}{\pi^2},$$

see [5, Equation (17.2.2) and Theorem 280], we obtain

$$\left| N_{p,a}(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll XY/Dp + Dp^{1/2}(\log p)^2 + \sum_{d > D} M_{p,a}(d; X, Y), \quad (5)$$

for every $a \in \mathcal{A}_p$.

We also remark that

$$\begin{aligned} \sum_{a=0}^{p-1} \sum_{d>D} M_{p,a}(d; X, Y) &= \sum_{d>D} \sum_{a=0}^{p-1} M_{p,a}(d; X, Y) \\ &= \sum_{d>D} \left\lfloor \frac{X}{d} \right\rfloor \left\lfloor \frac{Y}{d} \right\rfloor \leq XY \sum_{d>D} \frac{1}{d^2} \ll XY/D. \end{aligned} \quad (6)$$

Therefore, using the bounds (5) and (6), we obtain

$$\sum_{a \in \mathcal{A}_p} \left| N_{p,a}(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll XY/D + Dp^{3/2}(\log p)^2. \quad (7)$$

For $a \notin \mathcal{A}_p$ we estimate $N_{p,a}(X, Y)$ trivially as

$$N_{p,a}(X, Y) \leq \min\{X, Y\} \deg f \ll \sqrt{XY}.$$

Thus by Lemma 1,

$$\sum_{a \notin \mathcal{A}_p} \left| N_{p,a}(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll \max\{\sqrt{XY}, XY/p\} \ll \sqrt{XY}. \quad (8)$$

Combining (7) and (8) and taking $D = X^{1/2}Y^{1/2}p^{-3/4}(\log p)^{-1}$ we conclude the proof. \square

Corollary 3. *Let f be a polynomial with integer coefficients which is absolutely irreducible and of degree bigger than one. If $XY \geq p^{3/2}(\log p)^{2+\varepsilon}$ for some fixed $\varepsilon > 0$, then*

$$N_{p,a}(X, Y) = \left(\frac{6}{\pi^2} + o(1) \right) \frac{XY}{p}$$

for all but $o(p)$ values of $a = 0, \dots, p-1$.

4 Visible Points on Almost All Reductions

Throughout this section, the implied constants in the notations $A \ll B$ and $A = O(B)$ may depend on the coefficients of f .

To simplify notation we put

$$\mathcal{F}_p(X, Y) = \mathcal{F}_{p,0}(X, Y) \quad \text{and} \quad N_p(X, Y) = N_{p,0}(X, Y).$$

Theorem 4. *Let f be a polynomial with integer coefficients which is absolutely irreducible and of degree bigger than one. Then for real T , X and Y such that $T \geq 2 \max(X, Y)$, we have*

$$\sum_{T/2 \leq p \leq T} \left| N_p(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll X^{1/2} Y^{1/2} T^{3/4 + o(1)},$$

where the sum is taken over all primes p with $T/2 \leq p \leq T$.

Proof. It is enough to consider T large enough so that f remains absolutely irreducible and of degree bigger than one for all $p, T/2 \leq p \leq T$. As before we have

$$\left| N_p(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \ll XY/Dp + Dp^{1/2}(\log p)^2 + \sum_{d>D} M_p(d; X, Y). \quad (9)$$

where

$$M_p(d; X, Y) = \#\{(x, y) \in \mathcal{F}_p(X, Y) \mid \gcd(x, y) \equiv 0 \pmod{d}\}.$$

We also remark that

$$\begin{aligned} \sum_{T/2 \leq p \leq T} \sum_{d>D} M_p(d; X, Y) &= \sum_{d>D} \sum_{T/2 \leq p \leq T} M_p(d; X, Y) \\ &= \sum_{d>D} \sum_{1 \leq s \leq X/d} \sum_{1 \leq t \leq Y/d} \sum_{\substack{T/2 \leq p \leq T \\ p \mid f(ds, dy)}} 1. \end{aligned} \quad (10)$$

Let \mathcal{Z} be set of integer zeros of f in the relevant box, that is

$$\mathcal{Z} = \{(u, v) \in \mathbb{Z}^2 : 1 \leq x \leq X, 1 \leq y \leq Y, f(u, v) = 0\}.$$

It is easy to see that $\#\mathcal{Z} \ll \min(X, Y) \leq \sqrt{XY}$. Indeed, it is enough to notice that since $f(U, V)$ is absolutely irreducible, each specialization $g_y(U) = f(U, y)$ with $y \in \mathbb{Z}$ and $h_x(V) = f(x, V)$ with $x \in \mathbb{Z}$ is a nonzero polynomials in U and V , respectively. (Under extra, but generic, hypotheses, one can invoke Siegel's theorem, which gives $\#\mathcal{Z} = O(1)$ but this does not lead to an improvement in our final bound.) Denoting by $\tau(k)$ the number of integer divisors of a positive integer k , we see that for each $(u, v) \in \mathcal{Z}$ there are at most $\tau(u) = X^{o(1)}$ (see [5, Theorem 317]) pairs (d, s) of positive integers with

$ds = u$, after which there is at most one value of t . Thus for these triples (d, s, t) , we estimate the inner sum over p in (10) trivially as T .

To estimate the rest of the sums, as before, we denote by $\omega(k)$ the number of prime divisors of a positive integer k and note that $\omega(k) \ll \log k$. Thus for $(u, v) \notin \mathcal{Z}$ we can estimate the inner sum over p in (10) as $\omega(|f(ds, dy)|) = (XY)^{o(1)}$. Therefore

$$\begin{aligned} \sum_{T/2 \leq p \leq T} \sum_{d > D} M_p(d; X, Y) &\leq \sum_{d > D} \sum_{\substack{1 \leq s \leq X/d \\ 1 \leq t \leq Y/d \\ (ds, dt) \in \mathcal{Z}}} \sum_{T/2 \leq p \leq T} 1 + \sum_{d > D} \sum_{\substack{1 \leq s \leq X/d \\ 1 \leq t \leq Y/d \\ (ds, dt) \notin \mathcal{Z}}} \sum_{p | f(ds, dy)} 1 \\ &\leq \#\mathcal{Z} X^{o(1)} T + (XY)^{o(1)} \sum_{d > D} \sum_{\substack{1 \leq s \leq X/d \\ 1 \leq t \leq Y/d \\ (ds, dt) \notin \mathcal{Z}}} 1 \\ &= (XY)^{1/2+o(1)} T + (XY)^{1+o(1)} D^{-1}. \end{aligned}$$

We now put everything together getting

$$\begin{aligned} \sum_{T/2 \leq p \leq T} \left| N_p(X, Y) - \frac{6}{\pi^2} \cdot \frac{XY}{p} \right| \\ \ll XY/D \log T + DT^{3/2} (\log T)^2 + T(XY)^{1/2+o(1)} + (XY)^{1+o(1)} D^{-1}, \end{aligned}$$

and take $D = X^{1/2} Y^{1/2} T^{-3/4}$ getting the result. \square

Corollary 5. *Let f be a polynomial with integer coefficients which is absolutely irreducible and of degree bigger than one. If $XY \geq T^{3/2+\varepsilon}$ for some fixed $\varepsilon > 0$, that*

$$N_p(X, Y) = \left(\frac{6}{\pi^2} + o(1) \right) \frac{XY}{p}$$

for all but $o(T/\log T)$ primes $p \in [T/2, T]$.

5 Remarks

Certainly it would be interesting to obtain an asymptotic formula for $N_{p,a}(X, Y)$ which holds for every a . Even the case of $X = Y = p$ would be of interest. We remark that for the polynomial $f(U, V) = UV$ such an asymptotic formula is give in [8] and is nontrivial provided $XY \geq p^{3/2+\varepsilon}$ for some fixed

$\varepsilon > 0$. However the technique of [8] does not seem to apply to more general polynomials.

We remark that studying such special cases as visible points on the curves of the shape $f(U, V) = V - g(U)$ (corresponding to points a graph of a univariate polynomial) and $f(U, V) = V^2 - X^3 - rX - s$ (corresponding to points on an elliptic curve) is also of interest and may be more accessible than the general case.

Acknowledgements.

This work began during a pleasant visit by I. S. to University of Texas sponsored by NSF grant DMS-05-03804; the support and hospitality of this institution are gratefully acknowledged. During the preparation of this paper, I. S. was supported in part by ARC grant DP0556431.

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