

# A note on $(k, n)$ -arcs

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## Abstract

We construct  $(k, n)$ -arcs in  $PG(2, q)$  with  $k$  approximately  $q^2/d$  and  $n$  approximately  $q/d$  for each divisor  $d$  of  $q - 1$ .

*Key words:*

*1991 MSC:*

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## 1

We denote by  $PG(2, q)$  the projective plane over the finite field  $GF(q)$  of  $q$  elements. A  $(k, n)$ -arc in  $PG(2, q)$  is a subset of  $k$  points with at most  $n$  on a line. In the First Irsee Conference on Finite Geometry, R. Hill talked about  $(k, n)$ -arcs with  $n = (q - 1)/2$  and remarked that very little is known about  $(k, n)$ -arcs except in the extremal cases of  $n = 2, 3, q - 1, q$  (See [1] and [2]). The purpose of this note is to improve the knowledge of the intermediate cases by proving:

**Theorem 1** *Let  $q$  be a power of the prime  $p$ . For any divisor  $d$  of  $q - 1$  with  $2 < d < p$ , there exists a  $(k, n)$ -arc in  $PG(2, q)$  with*

$$k \geq (q^2 + q + 1)/d - (d - 2)\sqrt{q} - 11dq$$

and  $n \leq (q + 1)/d + (d - 2)\sqrt{q}$ .

**PROOF.** Let  $d$  be a divisor of  $q - 1$  and let  $f(x, y, z)$  be a homogeneous polynomial of degree  $d$  such that the curve  $X : f(x, y, z) = 0$  is smooth. For any coset  $C$  of the  $d$ -th powers in  $GF(q)^*$ , the set  $\{(x, y, z) \in PG(2, q) \mid f(x, y, z) \in C\}$

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$C\}$  is well-defined and the union of those sets is the complement of  $X$  in  $PG(2, q)$ . Therefore one of these sets has at least  $(q^2 + q + 1 - \#X(GF(q)))/d$  points. We select this set  $K$ , minus any lines it might contain, as our arc and the bound on  $k$  follows from the Weil bound applied to  $X$  once we show that  $K$  contains at most  $11d - 24$  lines.

Let  $L$  be a line in  $PG(2, q)$ , which we may assume by a change of coordinates to be  $z = 0$ . The number of points of the intersection of our set with  $L$  is the number of points  $(x, y)$  in  $PG(1, q)$  with  $f(x, y, 0)$  in the chosen coset. By choosing a coset representative  $c$  we want to count the number of solutions to  $f(x, y) = cw^d$  and note that a given  $(x, y)$  will lead to  $d$  values of  $w$ . The  $f(x, y, 0) = cw^d$  determines an algebraic curve and it is easy to check that it is either smooth or a union of  $d$  lines. If the curve is smooth then the bound on the number of points  $(x, y)$  follows from the Weil bound applied to this curve. If it is a union of lines, then  $L$  is contained in  $K$  but then it was removed in the original construction.

To show the upper bound on the lines we note that the lines correspond to sets of  $d$  lines on the surface  $f(x, y, z) = cw^d$ . Finally, the surface contains at most  $d(11d - 24)$  by the positive characteristic analogue of Salmon's theorem, proved in [4].

The construction of the theorem also works for  $d = 2$  provided that the surface  $f(x, y, z) = cw^2$  in the proof is an elliptic quadric so that it has no lines. the resulting arcs probably coincide with the Barlotti arcs (see [1]) but we have not checked this.

By taking a pencil of lines through a point of a  $(k, n)$ -arc we get that  $k \leq (n - 1)q + n$  and there are only small improvements known to this bound in general (see [2]). A way to produce  $(k, n)$ -arcs is to consider algebraic curves of degree  $n$  with no linear components. For small  $n$  those are usually complete as arcs but  $k$  is much smaller than the above bound. We note that, in [3], we constructed smooth algebraic curves of degree  $n$  with  $k = n(q + 1 + n)/2$  points for  $n = q - 1 - 2d$  for the divisors  $d < (q - 1)/2$  of  $q - 1$ . So these are somewhat large as  $(k, n)$ -arcs and it would be interesting to know whether they are complete.

## References

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