Conics over function fields and the Artin-Tate conjecture

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Abstract: We prove that the Hasse principle for conics over function fields is a simple consequence of a provable case of the Artin-Tate conjecture for surfaces over finite fields.

Hasse proved that a conic over a global field has a rational point if and only if it has points over all completions of the global field, an instance of the so-called local-global or Hasse principle. The case of the rational numbers is an old result of Legendre, who gave an elementary proof and a similar proof can be given in the case of rational functions over a finite field. Hasse's proof, on the other hand, is a consequence of a more general result in Class Field Theory, the Hasse norm theorem. One purpose of this paper is to give a new proof of the local-global principle for conics over function fields, as a relatively simple consequence of a provable case of the Artin-Tate conjecture for surfaces over finite fields. This proof might be more complicated overall, once the work on the Artin-Tate conjecture is factored in, but it might be worthwhile recording since it suggests a new approach to local-global principles which might work in other contexts, such as number fields or higher dimensions. We also deduce from the Artin-Tate conjecture, together with a recent result [LLR] on Brauer groups, the Hilbert reciprocity law. We also do a careful study of conic bundles over curves over finite fields which might have independent interest. We assume odd characteristic to simplify some calculations, but this assumption might be superfluous. We give a proof of the case we need of the Artin-Tate conjecture in the proof but it might well be among the previously known cases.

Theorem. *Let K be a function field in one variable over $\mathbf{F}_q$, the finite field of q elements, q odd, and C/K a conic, that is, a projective plane curve of degree 2 defined over K. For a place v of K, let K_v denote the completion of K at v, and assume that $C(K_v)$ is non-empty for all v. Then $C(K)$ is non-empty.*

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Proof: If C is reducible over K then it is a union of two lines and it clearly has points over K and over all K_v. If C is irreducible over K, but factors over an extension, then it is a union of two lines conjugate over K and their unique common point is defined over K and over all K_v. We can therefore reduce to the case where C is absolutely irreducible.

From the curve C/K we construct a surface $X/\mathbf{F}_q$ with a fibration $f : X \to B$ where $B/\mathbf{F}_q$ is a curve with function field K and such that the generic fiber of f is C. This surface X may be singular but by suitable resolution of singularities, we can replace it by another surface fibered over B, which is smooth. So, without loss of generality, we assume that X is smooth. The fibration f has a finite number of singular fibres at places $v_1, \ldots, v_r$ of K. Since we started with a fibration by conics, the singular fibers consist of a pair of lines, which might degenerate to a double line.

Let t be a local parameter at one of the places $v_1, \ldots, v_r$. We can write, after a change of coordinates in the plane, an equation for C of the form $ax^2 + by^2 = c$ where a, b, c are power series in t, not all vanishing at $t = 0$ (simply by scaling the coefficients) and none having a multiple zero at $t = 0$. Indeed, if $v(a) > 1$ then we can replace x by tx and a by a/t^2.

We now study the nature of the singular fiber in more detail. Note that we can permute a, b, c at will. If, say, a, c both vanish at $t = 0$, then the point on the fiber $t = 0$ with $y = 0$ and $a'(0)x^2 = c'(0)$ is a singular point of X. The blow-up consists of replacing y by ty and a new equation $a/tx^2 + tby^2 = c/t$ and only one coefficient now vanishes at $t = 0$. Returning to the original equation, if c vanishes at $t = 0$, there are two possibilities. First is that $-a(0)/b(0)$ is a square in the residue field, in which case the fiber at $t = 0$ is a pair of lines defined over the residue field. The second case is that $-a(0)/b(0)$ is not a square in the residue field. Then the fiber at $t = 0$ is a pair of lines defined over a quadratic extension of the residue field. A local point $x, y \in K_v$ would have to be given by power series with $x(0) = y(0) = 0$, since this is the only rational point on the fiber at $t = 0$, but then $ax^2 + by^2$ has at least a double zero at $t = 0$, whereas c by assumption has a simple zero. Therefore, this case does not happen under the assumption that C has local points

everywhere and all the singular fibers consist of a pair of distinct lines defined over the residue field.

We now compute the zeta function of $X/\mathbf{F}_q$. It is a product of Euler factors over the places of K and, for $v \neq v_1, \ldots, v_r$, this factor is the zeta function of $\mathbf{P}^1$ over the residue field of v evaluated at $t^{\deg v}$. For $v = v_1, \ldots, v_r$, this factor is the same as before times an extra $1/(1-(qt)^{\deg v})$. Thus, if $Z_B(t)$ denotes the zeta function of $B/\mathbf{F}_q$, the zeta function of $X/\mathbf{F}_q$ is

$$Z_X(t) = Z_B(t)Z_B(qt)\prod_{i=1}^{r} 1/(1-(qt)^{\deg v_i}).$$

The Artin-Tate conjecture for smooth, projective, irreducible surfaces over finite fields can be stated as follows. Write $Z_X(t) = P_1(t)P_3(t)/(1-t)P_2(t)(1-q^2t)$, where the P_i are polynomials with integer coefficients and $P_2(t) = (1-qt)^\rho R(t), R(1/q) \neq 0$. Then the conjecture states that ρ is the rank of $NS(X)$, the Néron-Severi group of X, and

$$R(1/q) = (-1)^{\rho-1}\det(D_i \cdot D_j)\#\operatorname{Br}(X)/q^{\alpha(X)}(\#NS(X)_{\mathrm{tor}})^2$$

where $D_1, \ldots, D_\rho$ is a basis of $NS(X)/NS(X)_{\mathrm{tor}}$, $\cdot$ is the intersection product, $\operatorname{Br}(X)$ is the Brauer group of X, and $\alpha(X)$ is a certain birational invariant of X, which in our case, since X is birational to a ruled surface over $\bar{\mathbf{F}}_q$, is zero.

In our case,

$$P_2(t) = (1-qt)^2\prod_{i=1}^{r}(1-(qt)^{\deg v_i})$$

so $\rho = r+2$ and $R(1/q) = \prod_{i=1}^{r}\deg v_i$.

We will see below that $r+2$ is indeed the rank of $NS(X)$ and by the results of [T] (completed by [M] to include the p-part) we infer that the full Artin-Tate conjecture is true for our X.

Let us now compute $NS(X)$ and the intersection pairing. First, we compute $NS(\bar{X})$, where $\bar{X}$ is the base change of X to $\bar{\mathbf{F}}_q$. In order to do that we need only to base change to some extension of $\mathbf{F}_q$, so for the moment assume that for X itself, there is a section of f (a section over $\bar{\mathbf{F}}_q$ exists by Tsen's theorem) and that $\deg v_i = 1$ for all i. Let S be a

section of f, F a smooth fiber of f and for $i = 1, \ldots, r$, let C_i be the component of the fiber above v_i with $C_i \cdot S = 0$. The existence of C_i follows since S is a section so meets each fiber at one point with multiplicity one, so can only meet one of the components of the fiber above v_i. As the general fiber of f has genus zero, any divisor on X is linear equivalent to a multiple of S plus a vertical divisor. As any smooth fiber is algebraically equivalent to F, any divisor on X is algebraically equivalent to a linear combination of S, F and a divisor supported above the fibers above the v_i and it immediately follows that $S, F, C_1, \ldots, C_r$ is a basis for $NS(X)$ and, in particular that, $NS(X)$ is torsion-free. It is also easy to compute the determinant of the intersection pairing since $S \cdot F = 1$, $C_i \cdot S = 0$, $F^2 = 0$, $F \cdot C_i = 0$ and $C_i^2 = -1$. Only the latter requires justification, but it follows since, denoting by C_i' the other component of the fiber above v_i, we have $(C_i + C_i')^2 = 0$ and $C_i \cdot C_i' = 1$, so, by symmetry, $C_i^2 = (C_i')^2 = -1$.

Recall that we assumed that our finite field was big enough so that $\deg v_i = 1$ and there was a section. Without this assumption, a similar argument goes through except that there may not be a section but $2S$ will be linearly equivalent to a rational divisor and that $\deg v_i$ may not be 1, in which case $C_i^2 = -\deg v_i$. Also, instead of taking F to be a fiber at a rational place (which may not exist) we take as F the pull-back by f of a divisor of degree one on B supported at places of good reduction. A simple calculation shows that the determinant of the intersection pairing of the set $2S, F, C_1, \ldots, C_r$ is $4(-1)^{r+1} \prod \deg v_i$. By an argument similar to above, $NS(X)$ is generated by $F, C_1, \ldots, C_r$ and either S or $2S$ so, in any case, its rank is $r+2$. As the Artin-Tate conjecture in our case gives that the determinant of the intersection pairing of a basis of $NS(X)$ has to be $(-1)^{r+1} \prod \deg v_i / (\# \operatorname{Br}(X))$, we conclude that $NS(X)$ is bigger than the span of $2S, F, C_1, \ldots, C_r$. But, if f has no section, $NS(X)$ cannot be bigger than this span, so we must conclude that f has a section, which proves our theorem and, as added bonus, that $S, F, C_1, \ldots, C_r$ is a basis for $NS(X)$ and that $\operatorname{Br}(X)$ is trivial.

Remark: If we do not assume that our conic is everywhere locally solvable then, in addition to the places $v_1, \ldots, v_r$, where the fiber is a pair of lines defined over the residue

field of the place, there will be places $w_1, \ldots, w_s$ where the fiber of f is a pair of lines defined and conjugate over a quadratic extension of the residue field. A simple calculation, similar to the above gives:

$$P_2(t) = (1-qt)^2 \prod_{i=1}^{r} (1-(qt)^{\deg v_i}) \prod_{j=1}^{s} (1+(qt)^{\deg w_j})$$

so $\rho = r+2$ and $R(1/q) = 2^s \prod_{i=1}^{r} \deg v_i$. Also, $NS(X)$ has as basis $2S, F, C_1, \ldots, C_r$, with the same notation as above, since f cannot have a section. It follows now that $\#\operatorname{Br}(X) = 2^{s-2}$. In the case that $B = \mathbf{P}^1$, Iskovskih ([I], corollary 2.7) proved that $\operatorname{Br}(X)$ is isomorphic to $(\mathbf{Z}/(2))^{s-2}$. In the general case, this is also true since it easy to see that all classes in $\operatorname{Br}(X)$ are split by a quadratic extension of $\mathbf{F}_q$ and thus have order 2. In [LLR] it is proved that $\#\operatorname{Br}(X)$ is a square and this implies that s is even, which proves Hilbert reciprocity.

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