

## On the geometry of spaces curves

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In this note we shall investigate the relationship between two maps attached to an algebraic curve in projective three space.

Let  $X$  be a reduced and irreducible complete algebraic curve in the projective space  $\mathbf{P}^3$  over the algebraically closed field  $k$ . Let  $X_0$  be the smooth part of  $X$  and  $CX$  be the conormal variety of  $X$  (see [1]) defined as the Zariski closure in  $\mathbf{P}^3 \times (\mathbf{P}^3)^\vee$  of  $\{(P, H) \mid P \in X_0, H \supseteq T_P X\}$ . Let also  $X^\vee$  be the dual variety of  $X$  defined as the natural projection in the dual projective space  $(\mathbf{P}^3)^\vee$  of  $CX$ . Denote by  $\pi'$  the natural map  $CX \rightarrow X^\vee$ .

Define also the Gauss map  $\gamma: X \rightarrow G$ , where  $G$  is the Grassmannian of lines in  $\mathbf{P}^3$ , by  $\gamma(P) = T_P X$  for  $P \in X$  and let  $\tilde{X}$  be the image of  $\gamma$ , and denote also by  $\gamma$  the map  $X \rightarrow \tilde{X}$ .

Denote by  $\deg_s \varphi$  (resp.  $\deg_i \varphi$ ) the separable (resp. inseparable) degree of a map  $\varphi$ . The following is known about the relationship between  $\gamma$  and  $\pi'$ .

**Theorem 1.** *We have that  $\deg_i \pi' = \deg_i \gamma$  and if  $\pi'$  is separable then  $\pi'$  is birational.*

The first part of the result follows from [1] theorem 3.5 and [2] proposition 4 and the remark after it. The second part is proved in [1] Theorem 3.5.

We have the following geometric interpretation for the separable degrees of  $\gamma$  and  $\pi'$  as it is easy to see. Firstly  $\deg_s \pi'$  is the number of points  $P \in X$  with  $T_P X \subseteq H$  where  $H$  is a generic element of  $X^\vee$  and secondly  $\deg_s \gamma$  is the number of points  $P \in X$  with  $T_P X = \ell$  where  $\ell$  is a generic element of  $\tilde{X}$ . From this and Theorem 1 it follows that  $\delta = \deg \pi' / \deg \gamma$  is the number of lines tangent to  $X$  contained in a generic element of  $X^\vee$ . We can now state our main result. A strange curve is a curve with all its tangents concurrent.

**Theorem 2.** *If  $X$  is not a strange curve then  $\delta = 1$ .*

**Proof:** Suppose  $X$  is a curve as above with  $\delta \geq 2$ . Fix  $P \in X_0$  and look at the planes  $H \supseteq T_P X$ . As  $H$  varies we produce  $P' \in X$  with  $T_P X \neq T_{P'} X$  and  $T_P X \subseteq H$  and the locus of the  $P'$  must be a curve, which is then the whole of  $X$  since  $X$  is irreducible. So, for any  $P' \in X_0$ ,  $T_P X, T_{P'} X$  are contained in a plane, hence meet. As  $P$  was arbitrary, we conclude that the tangents at any two smooth points of  $X_0$  meet.

Fix now  $P, P' \in X_0$  and take  $Q \in X_0$  generic. If  $T_Q X$  meets  $T_P X, T_{P'} X$  in distinct points then  $T_Q X$ , hence  $Q$ , is contained in the plane spanned by  $T_P X, T_{P'} X$  and it follows that  $X$  is a plane curve. On the other hand it is easy to see that, for a plane curve,  $\delta = 1$ . Thus  $T_Q X$  meets  $T_P X \cap T_{P'} X$  and  $X$  is a strange curve, as was to be proved.

We now analyse the case of a strange curve. Let  $C$  be the center of  $X$ , that is, the common point of its tangents. Then  $\tilde{X}$  is contained in the  $\mathbf{P}^2$  of lines passing through  $C$  and hence can be considered as a plane curve and, as it is easy to see,  $\delta = \deg \tilde{X}$ . As a line in the  $\mathbf{P}^2$  of lines passing through  $C$  correspond to the set of lines contained in a plane passing through  $C$  we conclude immediately that:

**Proposition.** *If  $X$  is a strange curve, not contained in a plane, then  $\delta \geq 2$ .*

**Example:** The following example shows that, for strange curves, there are no further constraints on the degrees of  $\pi'$  and  $\gamma$ . Let  $X$  be the projective closure of the affine curve given in parametric form by  $(t, t^{rq}, t^{\delta rq})$  where  $\delta \geq 2$  is arbitrary,  $q$  is a power of char  $\neq 0$  and char  $\nmid r$ . Then it is easy to see that  $X$  is a strange curve, has  $\deg \gamma = rq$ ,  $\deg_i \gamma = q$  and  $\delta = \deg \tilde{X}$ .

Thm. 2 also proved by Rathmann, Math. Ann 276 (1986) on pg 572

### References

- [1] Hefez, A. and Kleiman, S.L., "Notes on the duality of projective varieties" in, Geometry today, Arbarello et al eds., Birkhäuser (1985) 143- 183.
- [2] Hefez, A. and Voloch, J.F., "Frobenius non classical curves" Arch.Math. (Basel), to appear

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