CHAPTER IV

HEIGHT FUNCTIONS

1. Normalized Absolute Values

Let k be an algebraic number field of degree d over $\mathbb{Q}$ and let v be a place of k. Thus v is an equivalence class of nontrivial absolute values on k. Then there exists a unique place u of $\mathbb{Q}$ such that $v|u$. As before the places of $\mathbb{Q}$ are conveniently indexed by the set $\{\infty, 2, 3, 5, 7, \dots\}$. Thus we use u to denote either ∞ or a prime number p. If v is an archimedean place then $v|\infty$ and if v is a non-archimedean place then $v|p$ for some prime p. We write k_v for the completion of k at v and $\mathbb{Q}_u$ for the completion of $\mathbb{Q}$ at u. Then k_v is a local field and a finite extension of $\mathbb{Q}_u$. We write $d_v = [k_v : \mathbb{Q}_u]$ for the local degree at the place v. As u is uniquely determined by v, we will also write $\mathbb{Q}_v$ for the completion of $\mathbb{Q}$ at the place u such that $v|u$. Alternatively, $\mathbb{Q}_v$ is the closure of $\mathbb{Q}$ in k_v. We note the important identities

$$d = \sum_{v|u} d_v, \tag{1.1}$$

and, if α is in k,

$$\mathrm{Norm}_{k/\mathbb{Q}}(\alpha) = \prod_{v|u} \mathrm{Norm}_{k_v/\mathbb{Q}_u}(\alpha), \tag{1.2}$$

where the sum and product are over the set of all places v of k such that $v|u$.

Next we select two absolute values from the place v. The first is denoted by $\| \ \|_v$ and defined as follows:

(i) if $v|\infty$ then $\| \ \|_v$ is the unique absolute value on k_v that extends the usual absolute value on $\mathbb{Q}_\infty$,

(ii) if $v|p$ then $\| \ \|_v$ is the unique absolute value on k_v that extends the usual p-adic absolute value on $\mathbb{Q}_p$.

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The second absolute value is denoted by $|\ |_v$ and defined by $|x|_v = \|x\|_v^{d_v/d}$ for all x in k_v. If u is a place of $\mathbb{Q}$ then $\|\ \|_u = |\ |_u$. However, for a number field k of degree greater than 1 over $\mathbb{Q}$ the fraction d_v/d is often less than 1, and then $\|\ \|_v$ and $|\ |_v$ are distinct but equivalent absolute values on k_v. The absolute values $\|\ \|_v$ are usually more convenient for making calculations and estimates. For example, if v is archimedean then the triangle inequality using $\|\ \|_v$ is sharper than the triangle inequality using $|\ |_v$. The absolute values $|\ |_v$ are important because they appear in the product formula. Also, the absolute values $|\ |_v$ are used to define *absolute* height functions, that is, height functions that are well defined on the field $\overline{\mathbb{Q}}$ of all algebraic numbers.

If v is an archimedean place of k and $d_v = 1$, then the pair $(k_v, \|\ \|_v)$ is isometrically isomorphic to $(\mathbb{R}, \|\ \|_\infty)$. If $d_v = 2$ then $(k_v, \|\ \|_v)$ is isometrically isomorphic to $(\mathbb{C}, \|\ \|_\infty)$. If v is a non-archimedean place of k then $v|p$ for some prime number p, and $k_v/\mathbb{Q}_p$ is an extension of degree d_v. In both cases we know that $\|\ \|_v$ has a unique extension from the local field k_v to an algebraic closure $\overline{k_v}$. Then $\|\ \|_v$ has a further unique extension to a completion Ω_v of $\overline{k_v}$. Of course the same remarks apply to the absolute value $|\ |_v$, which also has a unique extension to $\overline{k_v}$, and then to Ω_v.

The simplest situation occurs if v is an archimedean place. Then $(\overline{k_v}, \|\ \|_v)$ is isometrically isomorphic to $(\mathbb{C}, \|\ \|_\infty)$ and so $\overline{k_v} = \Omega_v$.

If v is a non-archimedean place of k then $\overline{k_v}$ is algebraically closed but not complete, and Ω_v has infinite transcendence degree over $\overline{k_v}$. Of course Ω_v is complete with respect to the metric topology induced by the extended absolute value $\|\ \|_v$, and we proved in Theorem 7.1 of Chapter 3 that Ω_v is algebraically closed. Thus Ω_v is a non-archimedean analogue of the complex numbers.

If u is a place of $\mathbb{Q}$ then at each place v of k such that $v|u$ we have

$$\|x\|_v = \|\operatorname{Norm}_{k_v/\mathbb{Q}_u}(x)\|_u^{1/d_v} \quad \text{for } x \in k_v, \tag{1.3}$$

and therefore

$$|x|_v = |\operatorname{Norm}_{k_v/\mathbb{Q}_u}(x)|_u^{1/d} \quad \text{for } x \in k_v. \tag{1.4}$$

If α is in k then α is in k_v at each place v of k, and (1.2) implies that

$$\prod_{v|u} |\alpha|_v = |\operatorname{Norm}_{k/\mathbb{Q}}(\alpha)|_u^{1/d}. \tag{1.5}$$

We say that a property holds for *almost all* places v of k if it holds for all but finitely many places v. For example, if α belongs to $k^\times$ then $\operatorname{Norm}_{k/\mathbb{Q}}(\alpha)$ is a nonzero rational number, and therefore (1.5) implies that

$$\prod_{v|u} |\alpha|_v = 1 \tag{1.6}$$

for almost all places u of $\mathbb{Q}$. In fact rather more can be proved.

THEOREM 1.1 (THE PRODUCT FORMULA). *If α belongs to $k^\times$ then $|\alpha|_v = 1$ for almost all places v of k, and*

$$\prod_v |\alpha|_v = 1. \tag{1.7}$$

PROOF. Let

$$f_\alpha(x) = x^N + a_1 x^{N-1} + a_2 x^{N-2} + \cdots + a_N$$

in $\mathbb{Q}[x]$ be the minimal polynomial for α over $\mathbb{Q}$. Obviously the set of primes that occur in the numerator and denominator of the nonzero coefficients of f_α is a finite set. Thus we have

$$|f_\alpha|_p = \max\{1, |a_1|_p, |a_2|_p, \ldots, |a_N|_p\} = 1 \tag{1.8}$$

for almost all prime numbers p. If (1.8) holds for some prime p, then it follows that $|\alpha|_v \leq 1$ for all places v such that $v|p$. Then (1.6) implies that $|\alpha|_v = 1$ for all places v such that $v|p$. Thus we get $|\alpha|_v = 1$ for almost all places v of k.

By our previous remark only finitely many factors in the product on the left of (1.7) are different from 1. Thus there is no question of convergence. We combine (1.5) and the product formula in $\mathbb{Q}$. In this way we conclude that

$$\begin{aligned}\prod_v |\alpha|_v &= \prod_u \Big\{ \prod_{v|u} |\alpha|_v \Big\} \\ &= \Big\{ \prod_u |\operatorname{Norm}_{k/\mathbb{Q}}(\alpha)|_u \Big\}^{1/d} = 1,\end{aligned}$$

where $\prod_u$ denotes a product over all places u of $\mathbb{Q}$.

Let l/k be an extension of algebraic number fields. Then each place w of l determines a unique place v of k such that $w|v$ and l_w/k_v is a finite extension of local fields. As before, the degree of the extension l/k is the sum of the local degrees, or more precisely

$$[l : k] = \sum_{w|v} [l_w : k_v], \tag{1.9}$$

where the sum is over all places w of l such that $w|v$. If α is in l then, analogous to (1.2), we have

$$\operatorname{Norm}_{l/k}(\alpha) = \prod_{w|v} \operatorname{Norm}_{l_w/k_v}(\alpha). \tag{1.10}$$

Because $\| \ \|_w$ is an extension of $\| \ \|_v$, the analogue of (1.3) is

$$\|x\|_w = \| \operatorname{Norm}_{l_w/k_v}(x)\|_v^{1/[l_w:k_v]} \quad \text{for } x \in l_w. \tag{1.11}$$

Next we use (1.11) together with

$$[l_w : \mathbb{Q}_v] = [l_w : k_v][k_v : \mathbb{Q}_v] \quad \text{and} \quad [l : \mathbb{Q}] = [l : k][k : \mathbb{Q}].$$

In this way we obtain the identity

$$\begin{aligned} |x|_w &= \|x\|_w^{[l_w:\mathbb{Q}_v]/[l:\mathbb{Q}]} \\ &= \| \operatorname{Norm}_{l_w/k_v}(x)\|_v^{[k_v:\mathbb{Q}_v]/[l:\mathbb{Q}]} \\ &= | \operatorname{Norm}_{l_w/k_v}(x)|_v^{1/[l:k]} \quad \text{for } x \in l_w. \end{aligned} \tag{1.12}$$

Plainly (1.12) is the analogue of (1.4) for the extension l_w/k_v. Combining (1.10) and (1.12) we find that

$$\prod_{w|v} |\alpha|_w = | \operatorname{Norm}_{l/k}(\alpha)|_v^{1/[l:k]} \quad \text{for } \alpha \in l, \tag{1.13}$$

and this is the analogue of (1.5) for the relative extension l/k. If α belongs to k then $\operatorname{Norm}_{l/k}(\alpha) = \alpha^{[l:k]}$. In this case (1.13) becomes

$$\prod_{w|v} |\alpha|_w = |\alpha|_v \quad \text{for } \alpha \in k. \tag{1.14}$$

THEOREM 1.2. *Let α belong to $\overline{\mathbb{Q}}$ and let $f_\alpha(x)$ in $\mathbb{Q}[x]$ be the minimal polynomial for α over $\mathbb{Q}$. Then the following are equivalent:*

1. *the minimal polynomial $f_\alpha(x)$ belongs to $\mathbb{Z}[x]$,*
2. *there exists a monic polynomial $g(x)$ in $\mathbb{Z}[x]$ such that $g(\alpha) = 0$,*
3. *the ring $\mathbb{Z}[\alpha]$ is finitely generated as a $\mathbb{Z}$-module,*
4. *if α belongs to an algebraic number field k, and v is a non-archimedean place of k, then $|\alpha|_v \le 1$.*

PROOF. It is trivial that (1) implies (2).

Assume that (2) holds, and write

$$g(x) = x^M + b_1 x^{M-1} + b_2 x^{M-2} + \cdots + b_M,$$

where $b_1, b_2, \ldots, b_M$ are integers. Then the $\mathbb{Z}$-module generated by $\{1, \alpha, \alpha^2, \ldots, \alpha^{M-1}\}$ contains

$$\alpha^M = -b_1\alpha^{M-1} - b_2\alpha^{M-2} - \cdots - b_M.$$

Using induction on N, we find that the $\mathbb{Z}$-module generated by $\{1, \alpha, \alpha^2, \ldots, \alpha^{M-1}\}$ contains α^N for all $N \geq M$. This verifies (3).

Assume that (3) holds, and let v be a non-archimedean place of a number field k that contains α. Let $\mathbb{Z}[\alpha]$ be generated by $\beta_1, \beta_2, \ldots, \beta_L$. Then every element γ in $\mathbb{Z}[\alpha]$ can be written as

$$\gamma = c_1\beta_1 + c_2\beta_2 + \cdots + c_L\beta_L$$

with integers $c_1, c_2, \ldots, c_L$. It follows that

$$|\gamma|_v \leq \max\{|\beta_1|_v, |\beta_2|_v, \ldots, |\beta_L|_v\} = B$$

is bounded for all γ in $\mathbb{Z}[\alpha]$. In particular, we get

$$|\alpha^N|_v = |\alpha|_v^N \leq B$$

for each positive integer N, and we conclude that $|\alpha|_v \leq 1$.

Assume that (4) holds, and let

$$f_\alpha(x) = x^N + a_1x^{N-1} + a_2x^{N-2} + \cdots + a_N \tag{1.15}$$

in $\mathbb{Q}[x]$ be the minimal polynomial for α over $\mathbb{Q}$. Let k be the algebraic number field generated over $\mathbb{Q}$ by the roots of f_α. Write $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_N$ for the roots of f_α, so that

$$f_\alpha(x) = \prod_{n=1}^{N}(x - \alpha_n) \tag{1.16}$$

in $k[x]$. It follows that each coefficient a_n can be written as

$$a_n = (-1)^n \sum_{j_1<j_2<\cdots<j_n} \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_n}. \tag{1.17}$$

Then for each non-archimedean place v of k we get

$$|a_n|_v \leq \max\{|\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_n}|_v : j_1 < j_2 < \cdots < j_n\} \leq 1. \tag{1.18}$$

Finally, for each prime number p the identity (1.5) and (1.18) imply that

$$|a_n|_p = \prod_{v|p} |a_n|_v \leq 1.$$

That is, each rational coefficient a_n is an integer.

An algebraic number α in $\overline{\mathbb{Q}}$ is called an *algebraic integer* if it satisfies one, and therefore all, of the four conditions in the statement of Theorem 1.2. If both α and β are algebraic integers then it follows using (4) of Theorem 1.2 that $\alpha + \beta$ and $\alpha\beta$ are algebraic integers. Thus the set of all algebraic integers in $\overline{\mathbb{Q}}$ is a subring, and of course it contains the element 1. If k is a number field we often write O_k for the set of algebraic integers in k. In view of Theorem 1.2, we have

$$O_k = \{\alpha \in k : |\alpha|_v \leq 1 \text{ for all non-archimedean places } v \text{ of } k\}. \tag{1.19}$$

Then it follows from (1.19) that O_k is a subring of k that contains 1. For example, the subring of algebraic integers in $\mathbb{Q}$ is $\mathbb{Z}$.

THEOREM 1.3. *Let α belong to $\overline{\mathbb{Q}}$ and let $f_\alpha(x)$ in $\mathbb{Q}[x]$ be the minimal polynomial for α over $\mathbb{Q}$. Then the following are equivalent:*

(1) *the minimal polynomial $f_\alpha(x)$ belongs to $\mathbb{Z}[x]$ and $f_\alpha(0) = \pm 1$,*
(2) *if α belongs to an algebraic number field k, then α belongs to O_k and*

$$\mathrm{Norm}_{k/\mathbb{Q}}(\alpha) = \pm 1, \tag{1.21}$$

(3) *if α belongs to an algebraic number field k, and v is a non-archimedean place of k, then $|\alpha|_v = 1$,*
(4) *both α and α^{-1} are algebraic integers.*

PROOF. Assume that (1) holds and let α belong to an algebraic number field k. Then α belongs to O_k by Theorem 1.2, and (1.21) follows from (1.11) in Chapter 3.

Assume that (2) holds. By Theorem 1.2 we have $|\alpha|_v \le 1$ at each non-archimedean place v of k. And (1.5) implies that

$$\prod_{v|\infty} |\alpha|_v = |\,\mathrm{Norm}_{k/\mathbb{Q}}(\alpha)|_\infty = 1.$$

Clearly $\alpha \ne 0$. If $|\alpha|_{v_0} < 1$ at some non-archimedean place v_0 of k, then we would have

$$\prod_v |\alpha|_v < 1.$$

But this contradicts the product formula (1.7), and it follows that $|\alpha|_v = 1$ at each non-archimedean place v of k.

If (3) holds then (4) is obvious.

Assume that (4) holds, then the minimal polynomial $f_\alpha(x)$ of α over $\mathbb{Q}$ belongs to $\mathbb{Z}[x]$ by Theorem 1.2. Similarly, the minimal polynomial $f_{\alpha^{-1}}(x)$ of α^{-1} over $\mathbb{Q}$ belongs to $\mathbb{Z}[x]$. Suppose that $\deg f_\alpha = N$. As $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^{-1})$, we also have $\deg f_{\alpha^{-1}} = N$. We find that these polynomials are related by the identity

$$x^N f_\alpha\bigl(x^{-1}\bigr) = f_\alpha(0) f_{\alpha^{-1}}(x). \tag{1.22}$$

In particular, (1.22) implies that

$$1 = f_\alpha(0) f_{\alpha^{-1}}(0).$$

Because both $f_\alpha(0)$ and $f_{\alpha^{-1}}(0)$ are rational integers, we have $f_\alpha(0) = \pm 1$.

An algebraic number α in $\overline{\mathbb{Q}}$ is called an *algebraic unit* if it satisfies one, and therefore all, of the four conditions in the statement of Theorem 1.3. Clearly the set of algebraic units in $\overline{\mathbb{Q}}$ is a subgroup of the multiplicative group $\overline{\mathbb{Q}}^\times$. If k is a number field we write

$$U_k = \{\alpha \in k : |\alpha|_v = 1 \text{ for all non-archimedean places } v \text{ of } k\}. \tag{1.23}$$

for the multiplicative subgroup of units in the ring O_k. Of course the group of units in $\mathbb{Z}$ is $\{\pm 1\}$.

THEOREM 1.4. *Let α belong to the multiplicative group $\overline{\mathbb{Q}}^{\times}$. Then the following are equivalent*

(1) *α is a root of unity, that is, α is a torsion element in the group $\overline{\mathbb{Q}}^{\times}$,*
(2) *if α belongs to an algebraic number field k, then $|\alpha|_v = 1$ at every place v of k.*

PROOF. Suppose that α is a root of unity, so that $\alpha^m = 1$ for some positive integer m. If α belongs to an algebraic number field k, then

$$1 = |\alpha^m|_v = |\alpha|_v^m$$

at every place v of k. Thus we must have $|\alpha|_v = 1$ at every place v of k.

Assume that (2) holds. If l/k is an extension of algebraic number fields then it is clear that (2) also holds with k replaced by l. Let l be a splitting field for the minimal polynomial $f_\alpha(x)$ of α over $\mathbb{Q}$. Then l/k is an extension of algebraic number fields. Let the minimal polynomial $f_\alpha(x)$ be written as in (1.15), (1.16), and (1.17). Of course $f_\alpha(x)$ belongs to $\mathbb{Z}[x]$ by (2) and Theorem 1.2. Using (2) and (1.18) we find that

$$\|a_n\|_w \leq \sum_{j_1<j_2<\cdots<j_n} 1 \leq \binom{N}{n} \leq 2^N$$

at each archimedean place w of l. That is

$$\|a_n\|_\infty \leq 2^N \tag{1.21}$$

for each $n = 1, 2, \ldots, N$. Thus the minimal polynomial $f_\alpha(x)$ is contained in a finite collection of at most $\left(2^{2N}+1\right)^N$ polynomials in $\mathbb{Z}[x]$. It follows that there are at most finitely many elements β in the field k that satisfy the condition

$$|\beta|_v = 1 \quad \text{for all places } v \text{ of } k. \tag{1.23}$$

Because α satisfies (1.23) it is obvious that α^l satisfies (1.23) for every positive integer l. Thus we must have $\alpha^l = \alpha^m$ for distinct positive integers l and m. This shows that α is a root of unity.

2. The Galois Action on Places

In this section we assume that l/k be a Galois extension of algebraic number fields. Then we write $G = \mathrm{Aut}(l/k)$ for the Galois group. Thus each element of G is an automorphism $\sigma : l \to l$ that fixes k, $|G| = [l : k]$ and

$$l^G = \{\alpha \in l : \sigma\alpha = \alpha\} = k.$$

Let v be a place of k and w a place of l with $w|v$. Then each σ in G determines a map

$$\alpha \to \|\sigma^{-1}\alpha\|_w \tag{2.1}$$

that is easily seen to be an absolute value on l. Because σ fixes k, the absolute value (2.1) extends the absolute value $\| \ \|_v$ on k. Therefore (2.1) determines a unique place of l and we write σw for this place. Obviously we have $\sigma w|v$. Alternatively, σw is the unique place of l such that

$$\|\sigma^{-1}\alpha\|_w = \|\alpha\|_{\sigma w} \quad \text{for all } \alpha \in l. \tag{2.2}$$

Next we observe that for σ and τ in G and all α in l we have

$$\begin{aligned}\|\alpha\|_{(\sigma\tau)w} &= \|(\sigma\tau)^{-1}\alpha\|_w \\ &= \|\tau^{-1}\sigma^{-1}\alpha\|_w \\ &= \|\sigma^{-1}\alpha\|_{\tau w} \\ &= \|\alpha\|_{\sigma(\tau\alpha)}.\end{aligned}$$

This shows that $w \to \sigma w$ defines a group action of G on the set of all places w of l that satisfy $w|v$. That is, if we define

$$W_v = W_v(l/k) = \{w : w \text{ is a place of } k, \text{ and } w|v\},$$

then the Galois group $G = \mathrm{Aut}(l/k)$ acts on W_v.

If w belongs to W_v then the *stabilizer* or *decomposition group* of w is the subgroup

$$G_w = \{\sigma \in G : \sigma w = w\}.$$

We note the important identity

$$\begin{aligned}G_{\tau w} &= \{\sigma \in G : \sigma\tau w = \tau w\} \\ &= \{\sigma \in G : \tau^{-1}\sigma\tau \in G_w\} \\ &= \tau G_w \tau^{-1}.\end{aligned} \tag{2.3}$$

If σ is in G and α_1 and α_2 are in l, then from (2.2) we have

$$\|\sigma\alpha_1 - \sigma\alpha_2\|_{\sigma w} = \|\alpha_1 - \alpha_2\|_w.$$

Thus the automorphism

$$\sigma : (l, \| \ \|_w) \to (l, \| \ \|_{\sigma w})$$

is an isometry with respect to the metric topologies induced by $\| \ \|_w$ and $\| \ \|_{\sigma w}$. Obviously the same remark applies to

$$\sigma^{-1} : (l, \| \ \|_{\sigma w} \to (l, \| \ \|_w).$$

It follows that σ has a unique extension to a continuous automorphism

$$\overline{\sigma} : (l_w, \| \ \|_w) \to (l_{\sigma w}, \| \ \|_{\sigma w}). \tag{2.4}$$

As σ fixes k, it is clear that $\overline{\sigma}$ fixes the closure k_v of k in l_w. If σ belongs to the decomposition group G_w then $\overline{\sigma}$ is an element of the Galois group $\mathrm{Aut}(l_w/k_v)$ and so $\sigma \to \overline{\sigma}$ defines an injection of G_w into $\mathrm{Aut}(l_w/k_v)$.

THEOREM 2.1. *Let l/k be a Galois extension of algebraic number fields and v a place of k. For each place w in W_v the finite extension l_w/k_v is Galois and the map $\sigma \to \overline{\sigma}$ is an isomorphism of G_w onto the Galois group* $\mathrm{Aut}(l_w/k_v)$. *Moreover, $G = \mathrm{Aut}(l/k)$ acts transitively on the set $W_v = W_v(l/k)$.*

PROOF. At each place w in W_v we have

$$|G_w| \leq |\operatorname{Aut}(l_w/k_v)| \leq [l_w : k_v]. \tag{2.5}$$

Now select a place $\hat{w}$ in W_v and then let $\sigma_1, \sigma_2, \ldots \sigma_J$ be a complete set of distinct representatives for the left cosets of $G_{\hat{w}}$ in G. Then

$$\{\sigma_j \hat{w} : j = 1, 2, \ldots, J\}$$

is a collection of $J = [G : G_{\hat{w}}]$ distinct elements of W_v. In view of (1.6) and (2.3) we have

$$\begin{aligned} |G| = [G : G_{\hat{w}}]|G_{\hat{w}}| &= \sum_{j=1}^{J} |G_{\sigma_j \hat{w}}| \\ &\leq \sum_{j=1}^{J} [l_{\sigma_j \hat{w}} : k_v] \leq \sum_{w \in W_v} [l_w : k_v] = [l : k] = |G|. \end{aligned} \tag{2.6}$$

It follows that there is equality throughout the inequality of (2.6) This shows that there is equality in the inequalities (2.5) for each extension l_w/k_v. Hence these extensions are Galois and $\sigma \to \overline{\sigma}$ is always an isomorphism. The fact that there is equality in the second inequality of (2.6) verifies the assertion that G acts transitively on W_v.

COROLLARY 2.2. *Let l/k be a Galois extension of algebraic number fields and v a place of k. Then for each place w in W_v, for each σ in* $\mathrm{Aut}(l/k)$ *and each α in l, we have*

$$|\sigma^{-1}\alpha|_w = |\alpha|_{\sigma w}. \tag{2.7}$$

PROOF. By the theorem the two local degrees $[l_w : k_v]$ and $[l_{\sigma w} : k_v]$ are equal. Therefore the identity (2.7) follows immediately from (2.2).

3. The Absolute Weil Height

We return to the general situation in which l/k is an extension of algebraic number fields, but not necessarily a Galois extension, and v is place of k. It will be useful to have a slight modification of (1.11), namely,

$$\prod_{w|v} \max\{1, |\alpha|_w\} = \max\{1, |\alpha|_v\} \quad \text{for } \alpha \in k. \tag{3.1}$$

To verify (3.1), observe that if $|\alpha|_v \le 1$ then $|\alpha|_w \le 1$ for all places w of l such that $w|v$, and so (3.1) is trivial. If $1 < |\alpha|_v$ then $1 < |\alpha|_w$ for all places w of l such that $w|v$, and (3.1) follows from (1.11). In view of (3.1) we have

$$\prod_{w} \max\{1, |\alpha|_w\} = \prod_{v} \max\{1, |\alpha|_v\} \quad \text{for } \alpha \in k, \tag{3.2}$$

where the product on the left of (3.2) is over *all* places w of l, and the product on the right of (3.2) is over *all* places v of k. Therefore we define the *absolute Weil height* (or simply the *height*)

$$h : \overline{\mathbb{Q}} \to [1, \infty)$$

as follows: if α is an algebraic number we select an algebraic number field k that contains α, and then we define

$$h(\alpha) = \prod_{v} \max\{1, |\alpha|_v\}. \tag{3.3}$$

For example, we may select $k = \mathbb{Q}(\alpha)$. If l is an algebraic number field that also contains α, then l/k is an extension of number fields, and (3.2) shows that $h(\alpha)$ is well defined. In particular, the right hand side of (3.3) does not depend on the field k. That is, $h(\alpha)$ can be computed using (3.3) and any algebraic number field that contains α.

The function h defined by (3.3) is a *multiplicative* height. We will sometimes find it convenient to use the additive version, which is simply $\log h(\alpha)$ in our notation. Thus we have

$$\log h(\alpha) = \sum_{v} \log^+ |\alpha|_v$$

for all α in $\overline{\mathbb{Q}}$. If α is not zero then another expression for the height of α is

$$2 \log h(\alpha) = \sum_{v} \big| \log |\alpha|_v \big|_\infty. \tag{3.4}$$

This follows by adding together the two identities

$$\sum_{v} \big| \log |\alpha|_v \big|_\infty = \sum_{v} \big\{ \log^+ |\alpha|_v + \log^- |\alpha|_v \big\}$$

and

$$0 = \sum_v \log |\alpha|_v = \sum_v \big\{ \log^+ |\alpha|_v - \log^- |\alpha|_v \big\}.$$

As an immediate application of (3.4) we get

$$\log h(\alpha^m) = |m|_\infty \log h(\alpha) \quad \text{where } \alpha \neq 0 \text{ and } m \in \mathbb{Z}. \tag{3.5}$$

In particular we note the special case

$$h(\alpha^{-1}) = h(\alpha).$$

It is easy to compute the height of a rational number. Suppose, for example, that $\alpha = r/s$, r and s are integers, $\gcd(r,s) = 1$, and $1 \leq s$. Using the product formula in $\mathbb{Q}$ we find that

$$\begin{aligned} h(r/s) &= \prod_u \max\{1, |r/s|_u\} \\ &= \Big(\prod_u |s|_u\Big)\Big(\prod_u \max\{1, |r/s|_u\}\Big) \\ &= \prod_u \max\{|r|_u, |s|_u\}. \end{aligned}$$

Because of the condition $\gcd(r,s) = 1$, we have $\max\{|r|_p, |s|_p\} = 1$ at each finite place $u = p$. It follows that

$$h(r/s) = \max\{|r|_\infty, |s|_\infty\}. \tag{3.6}$$

Lemma 3.1. *If $\alpha_1, \alpha_2, \ldots, \alpha_N$ are algebraic numbers, then*

$$h(\alpha_1 + \alpha_2 + \cdots + \alpha_N) \leq N \prod_{n=1}^N h(\alpha_n), \tag{3.7}$$

and

$$h(\alpha_1 \alpha_2 \ldots \alpha_N) \leq \prod_{n=1}^N h(\alpha_n). \tag{3.8}$$

Proof. Let k be an algebraic number field of degree d over $\mathbb{Q}$ that contains all of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$. At each finite place v of k we have

$$\begin{aligned} \max\{1, |\alpha_1 + \alpha_2 + \cdots + \alpha_N|\}_v\} &\leq \max\{1, \max\big\{|\alpha_1|_v, |\alpha_2|_v, \ldots, |\alpha_N|_v\}\big\} \\ &\leq \prod_{n=1}^N \max\{1, |\alpha_n|_v\}. \end{aligned} \tag{3.9}$$

If v is an infinite place of k then

$$\begin{aligned}\max\{1, \|\alpha_1 + \alpha_2 + \cdots + \alpha_N\|_v\} &\leq \max\{1, \|\alpha_1\|_v + \|\alpha_2\|_v + \cdots + \|\alpha_N\|_v\} \\ &\leq N \prod_{n=1}^{N} \max\{1, \|\alpha_n\|_v\}.\end{aligned}$$

Therefore

$$\max\{1, |\alpha_1 + \alpha_2 + \cdots + \alpha_N|_v\} \leq N^{d_v/d} \prod_{n=1}^{N} \max\{1, |\alpha_n|_v\}, \tag{3.10}$$

where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. The bound (3.7) follows now using (1.1), (3.9) and (3.10).

The inequality (3.8) is an immediate consequence of the local inequalities

$$\max\{1, |\alpha_1\alpha_2 \ldots \alpha_N|_v\} \leq \prod_{n=1}^{N} \max\{1, |\alpha_n|_v\}$$

which plainly hold at each place v of k.

Lemma 3.2. *Assume that k be an algebraic number field of degree d over $\mathbb{Q}$ and let S be a subset of places of k. If α is an element of $k^\times$ then*

$$h(\alpha)^{-1} \leq \prod_{v \in S} |\alpha|_v \leq h(\alpha). \tag{3.11}$$

If α_1 and α_2 are distinct elements of k then

$$\big(2h(\alpha_1)h(\alpha_2)\big)^{-1} \leq \prod_{v \in S} |\alpha_1 - \alpha_2|_v \leq \big(2h(\alpha_1)h(\alpha_2)\big). \tag{3.12}$$

Proof. The upper bound in (3.11) is immediate from

$$\prod_{v \in S} |\alpha|_v \leq \prod_{v \in S} \max\{1, |\alpha|_v\} \leq h(\alpha).$$

Then the lower bound in (3.11) follows because

$$\prod_{v \in S} |\alpha^{-1}|_v \leq h(\alpha^{-1}) = h(\alpha).$$

Finally, (3.12) is a consequence of (3.7) and (3.11).

It is instructive to restrict the domain of the map $\alpha \to \log h(\alpha)$ to the multiplicative group $\overline{\mathbb{Q}}^\times$. Then we have

$$\log h : \overline{\mathbb{Q}}^\times \to [0, \infty).$$

It follows from Theorem 1.4 that

$$\left\{\alpha \in \overline{\mathbb{Q}}^\times : \log h(\alpha) = 0\right\} = \mathrm{Tor}\left(\overline{\mathbb{Q}}^\times\right)$$

is the torsion subgroup of $\overline{\mathbb{Q}}^\times$. From the definition (3.3) we find that

$$\log h(\alpha) = \log h(\alpha\zeta)$$

whenever α belongs to $\overline{\mathbb{Q}}^\times$ and ζ belongs to $\mathrm{Tor}\left(\overline{\mathbb{Q}}^\times\right)$. This shows that

$$\alpha \to \log h(\alpha) \tag{3.13}$$

is well defined on cosets of the quotient group $\overline{\mathbb{Q}}^\times/\mathrm{Tor}\left(\overline{\mathbb{Q}}^\times\right)$. Write $G = \overline{\mathbb{Q}}^\times/\mathrm{Tor}\left(\overline{\mathbb{Q}}^\times\right)$, so that $\log h : G \to [0, \infty)$ is well defined and satisfies $\log h(\alpha) = 0$ if and only if α is the identity element in G. Then it follows from (3.8) that the map

$$(\alpha, \beta) \to \log h(\beta^{-1}\alpha) \tag{3.14}$$

defines a metric on the group G. Obviously this metric induces a metric topology in G. By arguing in a manner similar to that which was used to establish the existence of completions, we can also show the existence of an Abelian topological group $\mathcal{G}$ such that $G \subseteq \mathcal{G}$ is a dense subgroup, the map (3.13) extends to a continuous map on $\mathcal{G}$, and the extended map (3.14) defines a metric on $\mathcal{G}$ that induces its metric topology.

4. Projective Heights

Let k be an algebraic number field of degree d over $\mathbb{Q}$. At each place v of k we write k_v for the completion of k at v and $d_v = [k_v : \mathbb{Q}_v]$ for the local degree. Our objective here is to define a system of norms on the finite dimensional vector spaces Ω_v^{N+1} for each positive integer N. As before, Ω_v is the completion of an algebraic closure $\overline{k_v}$ and we recall that Ω_v is both a complete metric space and an algebraically closed field. Let

$$\boldsymbol{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}$$

denote a (column) vector in Ω_v^{N+1}. Then we define $\| \ \|_v : \Omega_v^{N+1} \to [0, \infty)$ by

$$\|\boldsymbol{x}\|_v = \begin{cases} \max\limits_n \|x_n\|_v & \text{if } v \nmid \infty \\ \Big(\sum\limits_{n=0}^{N} \|x_n\|_v^2\Big)^{1/2} & \text{if } v|\infty. \end{cases} \tag{4.1}$$

It is obvious that $\| \ \|_v$ is a norm on Ω_v^{N+1} with respect to the absolute value $\| \ \|_v$. It will be convenient to define a second norm $| \ |_v : \Omega_v^{N+1} \to [0, \infty)$ with respect to the absolute value $| \ |_v$ by setting

$$|\boldsymbol{x}|_v = \big(\|\boldsymbol{x}\|_v\big)^{d_v/d}.$$

We have used the same notation for norms and absolute values, but this should not cause confusion because norms are applied to vectors and absolute values are applied to scalars.

If $\boldsymbol{\xi} \neq \mathbf{0}$ is a vector in $k^{N+1} \setminus \{\mathbf{0}\}$ then $\boldsymbol{\xi}$ belongs to $k_v^{N+1} \subseteq \Omega_v^{N+1}$ at each place v of k. As $\boldsymbol{\xi}$ has finitely many coordinates and at least one coordinate is nonzero, it is clear that $|\boldsymbol{\xi}|_v = 1$ at almost all places v. Therefore, we define the height function

$$H : k^{N+1} \setminus \{\mathbf{0}\} \to (0, \infty)$$

by

$$H(\boldsymbol{\xi}) = \prod_v |\boldsymbol{\xi}|_v. \tag{4.2}$$

If $\alpha \neq 0$ is in k then from the product formula we get the basic identity

$$H(\alpha\boldsymbol{\xi}) = \prod_v |\alpha\boldsymbol{\xi}|_v = \Big(\prod_v |\alpha|_v\Big)\Big(\prod_v |\boldsymbol{\xi}|_v\Big) = H(\boldsymbol{\xi}). \tag{4.3}$$

Of course (4.3) shows that H is well defined on the N-dimensional projective space $\mathbb{P}^N(k)$. Since we may select a representative of each vector in $\mathbb{P}^N(k)$ that has at least one homogeneous coordinate equal to 1, we also find that

$$H : \mathbb{P}^N(k) \to [1, \infty).$$

5. Heights on Polynomials

In this section we consider several functions that measure the size or complicatedness of a polynomial in different ways. We restrict our attention here to polynomials in one variable, but later we will consider similar questions for polynomials in several variables. To begin with we define a local measure of size at each place v of the number field k. Then we define corresponding global measures of complicatedness for polynomials with coefficients in k by taking a product over all the local measures. Because of the way we have normalized the absolute values $|\ |_v$, the global functions are absolute. That is, they do not depend on the choice of an algebraic number field within which these measures can be computed. Thus they are well defined on the polynomials in $\overline{\mathbb{Q}}(T)$.

Let v be a place of the algebraic number field k, and

$$f(T) = a_0T^L + a_1T^{L-1} + \dots a_{L-1}T + a_L, \quad \text{where } a_0 \neq 0, \tag{5.1}$$

a polynomial in the ring $\Omega_v[T]$ that is not identically zero. Because Ω_v is algebraically closed, f factors in $\Omega_v[T]$ as

$$f(T) = a_0 \prod_{l=1}^{L} (T - \alpha_l), \tag{5.2}$$

where $\alpha_1, \alpha_2, \dots, \alpha_L$ are the not necessarily distinct roots of f in Ω_v. We define the *local Mahler measure* of f by

$$\mu_v(f) = |a_0|_v \prod_{l=1}^{L} \max\{1, |\alpha_l|_v\}. \tag{5.3}$$

It follows immediately from (5.3) that the local Mahler measure is multiplicative. That is, if f and g are not identically zero polynomials in $\Omega_v[T]$, then

$$\mu_v(fg) = \mu_v(f)\mu_v(g). \tag{5.4}$$

Another obvious measure of the size of $f(T)$ is the norm of its vector of coefficients. We therefore define the *local height* of the polynomial $f(T)$ by

$$H_v(f) = \begin{cases} \max\limits_l |a_l|_v & \text{if } v \nmid \infty \\ \Big(\sum\limits_{l=0}^{L} \|a_l\|_v^2\Big)^{d_v/2d} & \text{if } v|\infty, \end{cases} \tag{5.5}$$

where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree and $d = [k : \mathbb{Q}]$ is the global degree. A further useful measure of size is the *local* sup-*norm* on the unit ball of Ω_v, which we define by

$$\nu_v(f) = \sup\{|f(z)|_v : z \in \Omega_v \text{ and } |z|_v \leq 1\}. \tag{5.6}$$

Finally, we will find it convenient to define

$$N(f) = \sum_{\substack{l=0 \\ a_l \neq 0}}^{L} 1, \tag{5.7}$$

so that $N(f)$ is the number of nonzero coefficients of f.

LEMMA 5.1. *Let f be a polynomial in $\Omega_v[T]$ that is not identically zero. If $v \nmid \infty$ then*

$$\mu_v(f) = H_v(f) = \nu_v(f), \tag{5.8}$$

and if $v|\infty$ then

$$\begin{aligned} \log H_v(f) - \frac{d_v}{d}\deg(f)\log 2 \le \log \mu_v(f) &\le \log H_v(f) \\ &\le \log \nu_v(f) \le \log H_v(f) + \frac{d_v}{2d}\log N(f). \end{aligned} \tag{5.9}$$

PROOF. Assume that $v \nmid \infty$. For $m = 0, 1, 2, \ldots, L$, let

$$e_m(t_1, t_2, \ldots, t_L) = \sum_{n_1 < n_2 < \cdots < n_m} t_{n_1} t_{n_2} \cdots t_{n_m}$$

denote the m-th elementary symmetric polynomial in variables $t_1, t_2, \ldots, t_L$. Then (5.2) implies that

$$f(T) = a_0 \sum_{l=0}^{L} (-1)^l e_l(\alpha_1, \alpha_2, \ldots, \alpha_L) T^{L-l}. \tag{5.10}$$

For each $m = 0, 1, 2, \ldots, L$ we have

$$\begin{aligned} |a_m|_v &= |a_0|_v |e_m(\alpha_1, \alpha_2, \ldots, \alpha_L)|_v \\ &\le |a_0|_v \max\{|\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_m}|_v : n_1 < n_2 < \cdots < n_m\} \\ &\le |a_0|_v \prod_{l=1}^{L} \max\{1, |\alpha_l|_v\} \\ &= \mu_v(f), \end{aligned}$$

and therefore $H_v(f) \le \mu_v(f)$. Next we define

$$\mathcal{R}_v = \{z \in \Omega_v : |z|_v \le 1\} \quad \text{and} \quad \mathcal{M}_v = \{z \in \Omega_v : |z|_v < 1\}.$$

Then $\mathcal{R}_v$ is an integral domain, $\mathcal{M}_v$ is the unique maximal ideal in $\mathcal{R}_v$, and the residue class field $\mathcal{R}_v/\mathcal{M}_v$ is infinite. As the set of roots of f in Ω_v is finite, there exists a unit ζ in $\mathcal{R}_v$ such that

$$|\zeta - \alpha_l|_v = \max\{1, |\alpha_l|_v\} \quad \text{for each } l = 1, 2, \ldots, L.$$

It follows that

$$\mu_v(f) \leq |f(\zeta)|_v \leq \nu_v(f).$$

Of course for any point z in $\mathcal{R}_v$ we have

$$\begin{aligned} |f(z)|_v &= |a_0 z^L + a_1 z^{L-1} + \ldots a_{L-1} z + a_L|_v \\ &\leq \max\{|a_0 z^L|_v, |a_1 z^{L-1}|_v, \ldots |z_L|_v\} \\ &\leq \max\{|a_l|_v : l = 0, 1, 2, \ldots, L\} \\ &= H_v(f), \end{aligned}$$

and this shows that $\nu_v(f) \leq H_v(f)$. Thus we have established the successive inequalities

$$H_v(f) \leq \mu_v(f) \leq \nu_v(f) \leq H_v(f),$$

and this proves (5.8).

For the remainder of the proof we assume that $v|\infty$. Then for each integer $m = 0, 1, 2, \ldots, L$ we have

$$\begin{aligned} \|e_m(\alpha_1, \alpha_2, \ldots, \alpha_L)\|_v &\leq \sum_{n_1 < n_2 < \cdots < n_m} \|\alpha_{n_1} \alpha_{n_2} \cdots \alpha_{n_m}\|_v \\ &\leq \binom{L}{m} \max\{\|\alpha_{n_1} \alpha_{n_2} \cdots \alpha_{n_m}\|_v : n_1 < n_2 < \cdots < n_m\} \\ &\leq \binom{L}{m} \prod_{l=1}^{L} \max\{1, \|\alpha_l\|_v\}. \end{aligned}$$

Using (5.10) we get

$$\begin{aligned} \sum_{l=0}^{L} \|a_l\|_v^2 &= \|a_0\|_v^2 \sum_{m=0}^{L} \|e_m(\alpha_1, \alpha_2, \ldots, \alpha_L)\|_v^2 \\ &\leq \Big(\sum_{m=0}^{L} \binom{L}{m}^2\Big) \|a_0\|_v^2 \prod_{l=1}^{L} \max\{1, \|\alpha\|_v^2\} \qquad (5.11) \\ &= \binom{2L}{L} \|a_0\|_v^2 \prod_{l=1}^{L} \max\{1, \|\alpha\|_v^2\}. \end{aligned}$$

and then (5.11) implies that

$$H_v(f) \le \binom{2L}{L}^{d_v/2d} \mu_v(f) \le 2^{d_v L/d} \mu_v(f),$$

and this verifies the first inequality on the left of (5.9). Next we write

$$\mathcal{U}_v = \{z \in \Omega_v : |z|_v = 1\}$$

for the compact, multiplicative group of units in Ω_v, and let λ_v denote a Haar measure on this group normalized so that $\lambda(\mathcal{U}_v) = 1$. From Jensen's formula, Jensen's inequality and Parseval's identity, we have

$$\begin{aligned} \log \|a_0\|_v + \sum_{l=1}^{L} \log^+ \|\alpha_l\|_v &= \int_{\mathcal{U}_v} \log \|f(z)\|_v \; d\lambda_v(z) \\ &\le \tfrac{1}{2} \log\Big(\int_{\mathcal{U}_v} \|f(z)\|_v^2 \; d\lambda_v(z) \Big) \\ &= \tfrac{1}{2} \log\Big(\sum_{l=0}^{L} \|a_l\|_v^2 \Big) \\ &\le \log\big(\sup\{\|f(z)\|_v : z \in \mathcal{U}_v\} \big). \end{aligned} \tag{5.12}$$

Thus (5.12) shows that

$$\log \mu_v(f) \le \log H_v(f) \le \log \nu_v(f).$$

The remaining inequality on the right of (5.9) follows from Cauchy's inequality:

$$\begin{aligned} \sup\{\|f(z)\|_v^2 : z \in \mathcal{U}_v\} &\le \Big(\sum_{l=0}^{L} \|a_l\|_v \Big)^2 \\ &\le N(f) \sum_{l=0}^{L} \|a_l\|_v^2, \end{aligned} \tag{5.13}$$

and then raising both sides of (5.13) to the power $d_v/2d$.

Again let f be a polynomial in $\Omega_v[T]$ given by (5.1) and (5.2). For each nonnegative integer n we write

$$D^{(n)} = (n!)^{-1} \Big(\frac{d}{dT} \Big)^n.$$

for the corresponding differential operator. And we note the simple identity

$$\{D^{(n)}f\}(T) = \sum_{l=n}^{L} a_l \binom{l}{n} T^{l-n}. \tag{5.14}$$

Now suppose that f is given by (5.1) and (5.2), and f has coefficients in k. Then f belongs to $\Omega_v[T]$ at all places v of k. As the set of coefficients of f is a finite set of algebraic numbers, it follows that $H_v(f) = 1$ for almost all v. Then (5.8) shows that $\mu_v(f) = \nu_v(f) = 1$ for almost all v. Therefore we define the *global Mahler measure* $\mu(f)$, the *global height* $H(f)$, and the *global* sup-*norm* $\nu(f)$ by

$$\mu(f) = \prod_v \mu_v(f), \quad H(f) = \prod_v H_v(f), \quad \text{and} \quad \nu(f) = \prod_v \nu_v(f), \tag{5.14}$$

respectively. Because of the way we have normalized the absolute values $\| \ \|_v$ and $| \ |_v$, the global functions μ, H and ν do not depend on the number field k that contains the coefficients of f. Thus we may regard them as positive real valued functions defined on the not identically zero polynomials in $\overline{\mathbb{Q}}[T]$. If α is a nonzero algebraic number then it follows from the product formula that

$$\mu(\alpha f) = \mu(f), \quad H(\alpha f) = H(f), \quad \text{and} \quad \nu(\alpha f) = \nu(f).$$

Hence each of the functions μ, H and ν is well defined on finite dimensional projective spaces over $\overline{\mathbb{Q}}$, where we identify a polynomial with its vector of coefficients.

Lemma 5.2. *Let f be a not identically zero polynomial in $\overline{\mathbb{Q}}[T]$. Then we have*

$$\log H(f) - \deg(f)\log 2 \le \log \mu(f) \le \log H(f) \le \log \nu(f) \le 2\log H(f). \tag{5.15}$$

Proof. By summing the inequalities in Lemma 5.1 over all places v of k, we find that

$$\log H(f) - \deg(f)\log 2 \le \log \mu(f) \le \log H(f) \le \log \nu(f) \le \log H(f) + \tfrac{1}{2}\log N(f).$$

To complete the proof we will show that

$$\tfrac{1}{2}\log N(f) \le \log H(f). \tag{5.16}$$

¿From the product formula we have

$$N(f) = \sum_{l=1}^{L}\Big(\prod_v |a_l|_v^2\Big) \le \Big\{\prod_{v\nmid\infty} H_v(f)^2\Big\}\Big\{\sum_{l=0}^{L}\Big(\prod_{v|\infty} |a_l|_v^2\Big)\Big\}. \tag{5.17}$$

Since $\sum_{v|\infty} d_v/d = 1$, we may use Hölders inequality to obtain

$$\begin{aligned}\sum_{l=0}^{L}\Big(\prod_{v|\infty}|a_l|_v^2\Big) &\le \prod_{v|\infty}\Big(\sum_{l=0}^{L}|a_l|_v^{2d/d_v}\Big)^{d_v/d}\\ &= \prod_{v|\infty}\Big(\sum_{l=1}^{L}\|a_l\|_v^2\Big)^{d_v/d} = \prod_{v|\infty}H_v(f)^2.\end{aligned} \tag{5.18}$$

Now (5.16) follows by combining (5.17) and (5.18).

If α is an algebraic number and k is an algebraic number field then there exists a unique monic, irreducible polynomial $f_\alpha(x)$ in $k[x]$ with $f_\alpha(\alpha) = 0$. Suppose that $\deg(f_\alpha) = L$, write $\alpha = \alpha_1$ and let $\alpha_2, \alpha_3, \ldots, \alpha_L$ denote the remaining roots of f_α in a splitting field l/k. In view of (2.7) we have

$$h(\alpha_1) = h(\alpha_2) = \cdots = h(\alpha_L).$$

Therefore the height of α and the global Mahler measure of f_α are connected by the basic identity:

$$\begin{aligned}\log\mu(f_\alpha) &= \sum_v \log\mu_v(f_\alpha) = \sum_v\sum_{l=1}^{L}\log^+|\alpha_l|_v\\ &= \sum_{l=1}^{L}\log h(\alpha_l) = \deg(f_\alpha)\log h(\alpha) = [k(\alpha):k]\log h(\alpha).\end{aligned} \tag{5.19}$$

References for Chapter 4

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