Stability for inverse point source problem

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1 Problem 1

Consider problem, without special notation, assume $\gamma$ and \( \Lambda \) as their usual definitions. The domain \( \Omega \subset \mathbb{R}^3 \).

\[
\Delta u + k^2 n(x)u = F \quad (1) \\
\gamma u = f \quad (2) \\
\Lambda \gamma u = g \quad (3)
\]

where the source \( F(x) = \sum_{j=1}^{m} P_j \delta(x - x_j) \), with \( P_j \in \mathbb{R}, x_j \in \Omega \), for convenience, \( n(x) \) is known as a smooth real function with compact support on \( \Omega \).

1.1 Stability of recovering location, respect to Cauchy data

What is the stability of recovering the location of \( x_j \). Suppose the number of point sources is known as \( m \). Or we formulate the stability argument as following statement.

**Statement 1.1** If \( u_l \) for \( l = 1, 2 \) be the solutions of equation (1) associated with Cauchy data \( (f_l, g_l) \) and sources \( F_l = \sum_{j=1}^{m} P_j \delta(x - x_j) \), if we have \( \| (f_1, g_1) - (f_2, g_2) \| \leq \epsilon \), can we find a permutation \( \pi \) of \( \{1, 2, \ldots, m\} \) such that

\[
\max_{j \geq 1} \| x_1^\pi - x_2^{\pi(j)} \| \leq \epsilon'
\]

and as \( \epsilon \to 0 \), \( \epsilon' \to 0 \) as well.

1.2 Construction with perturbation

Since the speed is known \( n(x) \), thus we set space \( \mathcal{M} = \{ v \in H^s, \Delta v + k^2 n v = 0 \} \), for any solution \( u \) satisfies $\Box$ we have

\[
\int v \Delta u + k^2 n u v dx = \sum_{j=1}^{m} P_j v(x_j) \quad (4) \\
\int u \Delta v + k^2 n u v dx = 0 \quad (5)
\]
Thus by Green’s formula, we have

$$\int_{\Gamma} \left( vg - f \frac{\partial v}{\partial n} \right) d\sigma = \sum_{j=1}^{m} P_j v(x_j) \quad (6)$$

And in order to find out the distance between points $x'_j$, we are going to generate a function which sits in $M$ and has zeros $x'_j$.

**Remark** Now our problem is to find a differentiable function $\phi$ such that $\phi \in M$ and $\phi(\xi_j) = 0$, where $j = 1, 2, \cdots, m$.

### 1.3 Existence by induction

Suppose $\phi$ satisfies that $\Delta \phi + k^2 n \phi = 0$, and $\phi(\xi_j) = 0$. Here $j = 1, \cdots, m$, where $m \geq 0$. We are going to find $u = \phi \phi$ such that $\Delta u + k^2 n u = 0$, where $\phi(\zeta) = 0, \zeta \neq \xi_j$, then

$$0 = \Delta (\phi \phi) + k^2 n \phi \phi = \phi \Delta \phi + 2 \nabla \phi \cdot \nabla \phi + \phi \Delta \phi + k^2 n \phi \phi = \phi \Delta \phi + 2 \nabla \phi \cdot \nabla \phi \quad (7)$$

We set out to find $\phi$ such that $\Delta \phi = 0$, and $\nabla \phi \cdot \nabla \phi = 0$.

#### 1.3.1 Moving frame

We consider local coordinate $\{e_1, e_2, e_3\}$, where $e_1 = \frac{\nabla \phi}{|\nabla \phi|}$, when $\nabla \phi \neq 0$, this defines a coordinate change $u = u(r)$. By vector analysis,

$$\Delta \phi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( h_2 h_3 \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( h_1 h_3 \frac{\partial \phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( h_1 h_2 \frac{\partial \phi}{\partial u_3} \right) \right] \quad (8)$$

where $h_1 = h_2 = h_3 = 1$ and $\frac{\partial \phi}{\partial u_1} = 0$, which simplifies the problem as

$$\Delta \phi = \left[ \left( \frac{\partial}{\partial u_2} \right)^2 + \left( \frac{\partial}{\partial u_3} \right)^2 \right] \phi = 0 \quad (9)$$

Any solution $\phi$ to this function will admit a solution to our original problem by setting $\tilde{\phi} = \phi(u) - \phi(u(\zeta))$. Here we can always select $\phi = u_2 + iu_3$.

#### 1.3.2 Existence

We have seen that if $m = 0$, we can always find a solution to the Helmholtz equation, then by induction, we can find the solution $\phi \in M$, such that $\phi(\xi_j) = 0$, for $j = 1, \cdots, m$.

Initially we take a $\Phi_0$, satisfies Helmholtz equation, then

$$\Delta \Phi_0 + k^2 n \Phi_0 = 0 \quad (10)$$

And $\nabla \Phi_0 \neq 0$ in $\Omega$. Apply the moving local coordinate, and take $\phi = u_2 + iu_3$.  

• For $\xi_1$, we take $\Phi_1(\xi) = \Phi_0(\xi) (\phi(\xi) - \phi(\xi_1))$, then $\langle \nabla \Phi_0, \nabla \phi \rangle = 0$ and $\Delta \phi = 0$.

• For $\xi_{k+1}$, we take $\Phi_{k+1}(\xi) = \Phi_k(\xi) (\phi(\xi) - \phi(\xi_{k+1}))$, we can see that

$$\nabla \Phi_k = \left[ \frac{\partial \Phi_k}{\partial u_1}, \frac{\partial \Phi_k}{\partial u_2}, i \frac{\partial \Phi_k}{\partial u_2} \right]^t \tag{11}$$

which means $\langle \nabla \Phi_k, \nabla \phi \rangle = 0$, $\Delta \phi = 0$. And the last thing is to check $\nabla \Phi_k \neq 0$, since

$$\frac{\partial \Phi_k}{\partial u_1} = \frac{\partial \Phi_0}{\partial u_1} \prod_{j=1}^k (\phi(\xi) - \phi(\xi_j)) \tag{13}$$

$$\frac{\partial \Phi_k}{\partial u_2} = \Phi_0 \sum_{j=1}^k \prod_{l=1, l \neq j}^k (\phi(\xi) - \phi(\xi_l)) \tag{14}$$

Since every root is simple, we know $\nabla \Phi_k$ is never going to be zero vector.

• Thus we take

$$\Phi(\xi) = \Phi_m(\xi) = \Phi_0(\xi) \prod_{j=1}^m (\phi(\xi) - \phi(\xi_j)) \tag{15}$$

which satisfies the Helmholtz equation and vanishes at $\xi_j$, $j = 1, \cdots, m$.

**Definition 1.2** We define projection map $S : \mathbb{R}^3 \to \mathbb{C}$, $S(\xi) = \phi(\xi)$, then our generated function is

$$\Phi = \Phi_0 \prod_{j=1}^m (S(\xi) - S(\xi_j)) \tag{16}$$

**Definition 1.3** We define function $\phi_l$ such that $\phi_l \in \mathcal{M}$ and $\phi_l$ vanishes at $x_1^j$ and $x_2^j$ except $x_1^2$.

1.4 Main proof

\begin{align*}
\int_{\Gamma} (g^1 \phi_l - f^1 \frac{\partial \phi_l}{\partial n}) \, d\sigma &= \sum_{j=1}^m P_j^1 \phi_l(x_j^1) = 0 \tag{17} \\
\int_{\Gamma} (g^2 \phi_l - f^2 \frac{\partial \phi_l}{\partial n}) \, d\sigma &= \sum_{j=1}^m P_j^2 \phi_l(x_j^2) = P_l^2 \phi_l(x_1^2) \tag{18}
\end{align*}
Subtract the above one from the below one,

\[ |\int_{\Gamma} ((g^2 - g^1)\phi_l - (f^2 - f^1) \frac{\partial \phi_l}{\partial n})d\sigma| \]

\[ = |P_l^2 \phi_l(x_l^2)| \tag{20} \]

\[ = |P_l^2 \prod_{j=1}^m (S(x_l^2) - S(x_j^1)) \prod_{j \neq l} (S(x_l^2) - S(x_l^1))\Phi_0(x_l^2)| \tag{21} \]

\[ \geq C|c_{\eta_m^l}^{m-1}| \tag{22} \]

\[ \geq C|c_{\xi_m^l}^{m-1}| \tag{23} \]

where \( c = \min P_l^2, \eta_l = \min_j |S(x_l^2) - S(x_j^1)|, \xi_l = \min_{i \neq l} |S(x_l^2) - S(x_i^2)| \)

While by Cauchy-Schwarz inequality.

\[ |\int_{\Gamma} ((g^2 - g^1)\phi_l - (f^2 - f^1) \frac{\partial \phi_l}{\partial n})d\sigma| \leq (\|g^2 - g^1\|\|\phi_l\| + \|f^2 - f^1\|\|\frac{\partial \phi_l}{\partial n}\|) \tag{24} \]

\[ \leq C\sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1}, \|\frac{\partial \phi_l}{\partial n}\|_{L^2(\Gamma)} \leq C_1 \sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1}. \tag{25} \]

And \( \|\phi_l\|_{L^2(\Gamma)} \leq C\sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1}, \|\phi_l\|_{L^2(\Gamma)} \leq C_1 \sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1}. \tag{25} \]

For each \( l \), we have

\[ |c_{\eta_l}^{m-1} e^{-kT}| \leq C\{\|g^2 - g^1\| + \|f^2 - f^1\|\} \sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1} \tag{26} \]

Thus,

\[ \max_i \eta_l \leq \left\{ \frac{C\sqrt{|\Gamma|}((\text{diam}(\Omega))^{2m-1}}{c_{\xi_l}^{m-1}} \{\|g^2 - g^1\| + \|f^2 - f^1\|\} \right\}^{\frac{1}{m}} \tag{27} \]

1.5 Stability of intensity, respect to Cauchy data

We set out to find a two-step stability for intensities. Suppose we have reconstructed the locations, and intensities are remained unknown. The problem is formulated as following.

**Statement 1.4** Suppose our sources are \( F^1 = \sum_{j=1}^m P_l^j \delta(x - x_j^1) \) and \( F^2 = \sum_{j=1}^m P_l^j \delta(x - x_j^2) \) associated with Cauchy data \((f^1, g^1)\) and with the knowledge of the maximal error on local locations of point sources, we are going to find out the stability on intensities w.r.t perturbation on Cauchy data.

Take

\[ H_l = \prod_{j \neq l} (S(x_j^1) - S(x_l^1)) \prod_{j \neq l} (S(x_j^1) - S(x_l^1)) \tag{28} \]

then according our construction, there is a \( \Phi_0 : \mathbb{R}^3 \rightarrow \mathbb{C} \) such that \( u_l = H_l \Phi_0 \) satisfies the Helmholtz equation:

\[ \Delta u_l + k^2 u_l = 0 \tag{29} \]
Similarly, we have
\[
\int_{\Gamma} (g^1 u^i - f^1 \frac{\partial u^i}{\partial n}) d\sigma = \sum_{j=1}^{m} P^1_j u^i(x^1_j) = P^1_i u^i(x^1_i)
\] (30)
\[
\int_{\Gamma} (g^2 u^i - f^2 \frac{\partial u^i}{\partial n}) d\sigma = \sum_{j=1}^{m} P^2_j u^i(x^2_j) = P^2_i u^i(x^2_i)
\] (31)

Therefore
\[
P^1_i u^i(x^1_i) - P^2_i u^i(x^2_i) = \int_{\Gamma} ((g^1 - g^2) u^i - (f^1 - f^2) \frac{\partial u^i}{\partial n}) d\sigma
\] (32)

The RHS can be bounded by Hölder estimation.
\[
\left| \int_{\Gamma} (g^1 - g^2) u^i - (f^1 - f^2) \frac{\partial u^i}{\partial n} d\sigma \right| \leq C \left( \|g^1 - g^2\| + \|f^1 - f^2\| \right) \sqrt{|\Omega|} (\text{diam}(\Omega))^{2m-2}
\] (33)

And LHS can be written as
\[
|P^1_i u^i(x^1_i) - P^2_i u^i(x^2_i)| = |(P^1_i - P^2_i) u^i(x^1_i)| + |P^2_i(u^i(x^1_i) - u^i(x^2_i))| \leq |(P^1_i - P^2_i) u^i(x^1_i)| - |P^2_i(u^i(x^1_i) - u^i(x^2_i))| \leq |(P^1_i - P^2_i) u^i(x^1_i)| - |P^2_i(u^i(x^1_i) - u^i(x^2_i))|
\] (34)
(35)
(36)

Observe \([33]\) and \([36]\) we can see
\[
|P^1_i - P^2_i| |u^i(x^1_i)| \leq C \left( \|g^1 - g^2\| + \|f^1 - f^2\| \right) \sqrt{|\Omega|} (\text{diam}(\Omega))^{2m-2} + \|P^2_i(u^i(x^1_i) - u^i(x^2_i))|
\]
\[
|P^1_i - P^2_i| \leq C \left( \|g^1 - g^2\| + \|f^1 - f^2\| \right) \sqrt{|\Omega|} (\text{diam}(\Omega))^{2m-2} + \beta M \eta
\]

where \(\beta = \max_{i=1}^{m} |P^2_i|, M = \max_{\Omega} |\nabla u|, \eta = \max_{\tau} \eta \) is defined in \([27]\). And
\[
|P^1_i - P^2_i| |u^i(x^1_i)| \geq C \|P^1_i - P^2_i\|_X \left( \text{diam}(\Omega) \right)^{m-1}
\] (37)

where \(\xi = \min_{j \neq i} |S(x^1_i) - S(x^1_j)|, \gamma_i = \min_{j \neq i} |S(x^1_i) - S(x^2_j)|\).

Thus
\[
|P^2_i - P^1_i| \leq \frac{C \left( \|g^1 - g^2\| + \|f^1 - f^2\| \right) \sqrt{|\Omega|} (\text{diam}(\Omega))^{2m-2} + \beta M \eta}{C \xi^{m-1} \gamma_i^{m-1}}
\] (38)

Remark: Here we can see that \(\gamma_i = \min_{j \neq i} |S(x^1_i) - S(x^2_j)| \geq \min_{j=1}^{m} |S(x^1_j) - S(x^2_j)| = \eta \)

### 2 Problem 2

2.1 Stability, respect to \(n(x)\)

We expect the stability as the following informal statement.
Statement 2.1 Consider fixed Cauchy data \((f, g)\) on the boundary, and refractive index \(n_l\), \(l = 1, 2\), the associated point sources are \(F_l = \sum_{j=1}^{m} P_{lj} \delta(x - x_{lj})\). If there is small difference (under some norm) on \(n_1, n_2\), we need to find out if there is small difference between the sources.

\[
\Delta u_1 + k^2 n_1 u_1 = \sum_{j=1}^{m} P_{lj} \delta(x - x_{lj}) \tag{39}
\]

\[
\Delta u_2 + k^2 n_2 u_2 = \sum_{j=1}^{m} P_{lj} \delta(x - x_{lj}) \tag{40}
\]

If we consider any \(\phi\) such that

\[
\Delta \phi + k^2 n_1 \phi = 0 \tag{41}
\]

Then for \(u_1\),

\[
\int_{\Gamma} \left( \phi g - f \frac{\partial \phi}{\partial n} \right) \, d\sigma = \sum_{j=1}^{m} P_{lj} \phi(x_{lj}) \tag{42}
\]

and for \(u_2\),

\[
\int_{\Gamma} \left( \phi g - f \frac{\partial \phi}{\partial n} \right) \, d\sigma = \sum_{j=1}^{m} P_{lj} \phi(x_{lj}) + \int_{\Omega} k^2 (n_1 - n_2) u_2 \phi \tag{43}
\]

subtract the above one from the below one,

\[
\sum_{j=1}^{m} P_{lj} \phi(x_{lj}) + \int_{\Omega} k^2 (n_1 - n_2) u_2 \phi = \sum_{j=1}^{m} P_{lj} \phi(x_{lj}) \tag{44}
\]

- We choose \(\phi = \phi_l\) to vanish at \(x_{lj}\) and \(x_{lj}\) except \(x_{lj}\), then we can get a similar result on location’s stability.
- Suppose we have been aware of the approximated locations, we take \(u_l\) to vanish at \(x_{lj}\) and \(x_{lj}\) except \(x_{lj}\) and \(x_{lj}\). Then we can get a similar result about the stability of intensity.