

# SPDE and portfolio choice

(joint work with M. Musiela)

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Thaleia Zariphopoulou  
The University of Texas at Austin

## Performance measurement of investment strategies



# Market environment

## Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$  ;  $W = (W^1, \dots, W^d)$  standard Brownian Motion

- Traded securities

$$1 \leq i \leq k \quad \begin{cases} dS_t^i = S_t^i \left( \mu_t^i dt + \sigma_t^i \cdot dW_t \right) , & S_0^i > 0 \\ dB_t = r_t B_t dt , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}, \sigma_t^i \in \mathbb{R}^d$  bounded and  $\mathcal{F}_t$ -measurable stochastic processes

- Postulate existence of an  $\mathcal{F}_t$ -measurable stochastic process  $\lambda_t \in \mathbb{R}^d$  satisfying

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

## Market environment

- Self-financing investment strategies  $\pi_t^0, \pi_t = (\pi_t^1, \dots, \pi_t^i, \dots, \pi_t^k)$
- Present value of this allocation

$$X_t = \sum_{i=0}^k \pi_t^i$$

$$dX_t = \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot (\lambda_t dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t dt + dW_t)$$

## Traditional framework

A (deterministic) utility datum  $u_T(x)$  is assigned at the **end** of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

## Backwards in time generation of the indirect utility

$$V(x, s; T) = \sup_{\pi} E_{\mathbb{P}}(u_T(X_T^{\pi}) | \mathcal{F}_s; X_s^{\pi} = x)$$

$$V(x, s; T) = \sup_{\pi} E_{\mathbb{P}}(V(X_t^{\pi}, t; T) | \mathcal{F}_s; X_s^{\pi} = x) \quad (\text{DPP})$$

$$V(x, s; T) = E_{\mathbb{P}}(V(X_t^{\pi^*}, t; T) | \mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility  
for all  $t \in [0, T)$



## Study of the value function process

- “Arbitrary” environments

Duality methods

Martingale representation results

- Markovian environments

HJB equation

Feedback optimal controls

Weak solutions

⋮

## A stochastic PDE for the value function process



## Intuition

- Assume that, for  $t \in [0, T]$ , the value function  $V(x, t)$  solves

$$dV(x, t) = b(x, t) dt + a(x, t) \cdot dW_t$$

where  $b, a$  are  $\mathcal{F}_t$ -measurable processes.

- Recall that for an arbitrary admissible portfolio  $\pi$ , the associated wealth process,  $X^\pi$ , solves

$$dX_t^\pi = \sigma_t \pi_t (\lambda_t dt + dW_t)$$

- Applying the Ito-Ventzell formula to  $V(X_t^\pi, t)$  yields

$$\begin{aligned} dV(X_t^\pi, t) &= b(X_t^\pi, t) dt + a(X_t^\pi, t) \cdot dW_t \\ &+ V_x(X_t^\pi, t) dX_t^\pi + \frac{1}{2} V_{xx}(X_t^\pi, t) d\langle X^\pi \rangle_t + a_x(X_t^\pi, t) \cdot d\langle W, X^\pi \rangle_t \\ &= \left( b(X_t^\pi, t) + V_x(X_t^\pi, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x(X_t^\pi, t) + \frac{1}{2} V_{xx}(X_t^\pi, t) |\sigma_t \pi_t|^2 \right) dt \\ &\quad + (a(X_t^\pi, t) + V_x(X_t^\pi, t) \sigma_t \pi_t) \cdot dW_t \end{aligned}$$

## Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity

$$\sup_{\pi} \left( V_x (X_t^{\pi}, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_t^{\pi}, t) + \frac{1}{2} V_{xx} (X_t^{\pi}, t) |\sigma_t \pi_t|^2 \right)$$

is well defined

- Calculating the optimum  $\pi^*$  yields

$$\pi_t^* = -\sigma_t^+ \frac{V_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t)}{V_{xx} (X_t^{\pi^*}, t)}$$

- Deduce that the above supremum is given by

$$M^* (X_t^{\pi^*}, t) = - \frac{|\sigma_t \sigma_t^+ (V_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t))|^2}{2V_{xx} (X_t^{\pi^*}, t)}$$

- The drift coefficient  $b$  must satisfy

$$b (X_t^{\pi^*}, t) = -M^* (X_t^{\pi^*}, t)$$

## SPDE for the value function process

- Market  $(\sigma_t, \sigma_t^+, \lambda_t)$ ; volatility  $a(x, t) \in \mathcal{F}_t$

$$dV = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(V\lambda + a)|^2}{\mathcal{A}^2 V} dt + a \cdot dW$$

$$V(x, T) = u_T(x) \in \mathcal{F}_0 ; \quad \mathcal{A} = \frac{\partial}{\partial x}$$

- Feedback optimal portfolio vector

$$\pi_t^* = \pi^*(X_t^{\pi, *}, t) = -\sigma^+ \frac{\mathcal{A}(V\lambda + a)}{\mathcal{A}^2 V}(X_t^{\pi, *}, t)$$

- Choices for the volatility process  $a$ ?

## A Markovian example

- $r_t = r(Y_t), \quad \mu_t = \mu(Y_t), \quad \sigma_t = \sigma(Y_t)$

$$dY_t = \theta(Y_t) dt + \Theta^T(Y_t) dW_t$$

- Value function

$$v(x, y, t; T) = \sup_{\pi} E (u_T(X_T^{\pi}) \mid X_t^{\pi} = x, Y_t = y)$$

- HJB equation

$$\begin{aligned} v_t + \sup_{\pi} \left( \frac{1}{2} |\sigma \pi|^2 v_{xx} + \sigma \pi \cdot \sigma \sigma^+ (\lambda v_x + \Theta v_{xy}) + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y \right) \\ = v_t - \frac{1}{2} \frac{|\sigma \sigma^+ (v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y = 0 \end{aligned}$$

- The SPDE for the value function process

$$V(x, t) = v(x, Y_t, t; T)$$

$$dV(x, t) = v_t dt + v_y \cdot dY + \frac{1}{2} v_{yy} \cdot d\langle Y \rangle$$

$$\stackrel{\text{HJB}}{=} \left( \frac{1}{2} \frac{|\sigma\sigma^+(v_x\lambda + \Theta v_{xy})|^2}{v_{xx}} - \frac{1}{2} \Theta^T \Theta \cdot v_{yy} - \theta \cdot v_y \right) dt$$

$$+ v_y \cdot (\theta dt + \Theta^T dW) + \frac{1}{2} v_{yy} \cdot \Theta^T \Theta dt$$

$$= \frac{1}{2} \frac{|\sigma\sigma^+(V_x\lambda + a_x(x, t))|^2}{V_{xx}} dt + a(x, t) \cdot dW$$

- The volatility process is **uniquely** determined:  $a(x, t) = \Theta v_y(x, Y_t, t; T)$

## Going beyond the deterministic terminal utility problem



## Motivation (partial)

- Terminal utility might be  $\omega$ -dependent

Liability management, indifference valuation

$$u_T(x, \omega) = -e^{-\gamma(x - C_T(\omega))} ; \quad C_T \in \mathcal{F}_T$$

Numeraire consistency

$$u_T(x, \omega) = -e^{-\gamma_T(\omega)x}$$

- Need to extend the value function process beyond  $T$
- Need to manage liabilities of arbitrary maturities

**How do we formulate investment performance criteria?**

## Investment performance process

$U(x, t)$  is an  $\mathcal{F}_t$ -adapted process,  $t \geq 0$

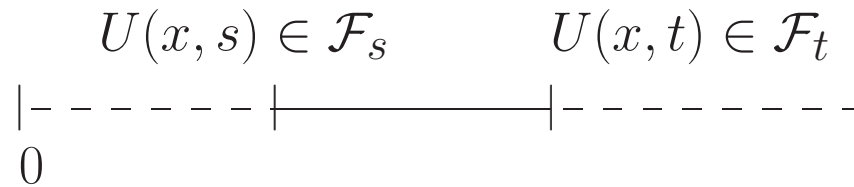
- The mapping  $x \rightarrow U(x, t)$  is increasing and concave
- For each self-financing strategy, represented by  $\pi$ , the associated (discounted) wealth  $X_t^\pi$  satisfies

$$E_{\mathbb{P}}(U(X_t^\pi, t) \mid \mathcal{F}_s) \leq U(X_s^\pi, s), \quad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by  $\pi^*$ , for which the associated (discounted) wealth  $X_t^{\pi^*}$  satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \quad 0 \leq s \leq t$$

## Optimality across times



$$U(x, s) = \sup_{\mathcal{A}} E(U(X_t^\pi, t) | \mathcal{F}_s, X_s = x)$$

- What is the meaning of this process?
- Does such a process always exist?
- Is it unique?

## Forward performance process

A datum  $u_0(x)$  is assigned at the beginning of the trading horizon,  $t = 0$

$$U(x, 0) = u_0(x)$$

## Forward in time criteria

$$E_{\mathbb{P}}(U(X_t^{\pi}, t) | \mathcal{F}_s) \leq U(X_s^{\pi}, s), \quad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) | \mathcal{F}_s) = U(X_s^{\pi^*}, s), \quad 0 \leq s \leq t$$

Many difficulties due to “inverse in time”

nature of the problem

## The forward performance SPDE



## The forward performance SPDE

Let  $U(x, t)$  be an  $\mathcal{F}_t$ -measurable process such that the mapping  $x \rightarrow U(x, t)$  is increasing and concave. Let also  $U = U(x, t)$  be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(U\lambda + a)|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

where  $a = a(x, t)$  is an  $\mathcal{F}_t$ -adapted process, while  $\mathcal{A} = \frac{\partial}{\partial x}$ .

Then  $U(x, t)$  is a forward performance process.

The process  $a$  may depend on  $t, x, U$ , its spatial derivatives etc.

## Optimal portfolios and wealth

### At the optimum

- The optimal portfolio vector  $\pi^*$  is given in the feedback form

$$\pi_t^* = \pi^*(X_t^*, t) = -\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t)$$

- The optimal wealth process  $X^*$  solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t) (\lambda dt + dW_t)$$

## Solutions to the forward performance SPDE

$$dU = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(U\lambda + a)|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

Local differential coefficients

$$a(x, t) = F(x, t, U(x, t), U_x(x, t))$$

### Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on  $U_x$  and  $U_{xx}$
- The equation is not (degenerate) elliptic

## Examples



## Choices of volatility coefficient

- The zero volatility case:  $a(x, t) = 0$

The forward performance SPDE simplifies to

$$dU = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}(U\lambda)|^2}{\mathcal{A}^2 U} dt$$

The process

$$U(x, t) = u(x, A_t) \quad \text{with} \quad A_t = \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds$$

with  $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ , increasing and concave with respect to  $x$ , and solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)

- $a(x, t) = 0$

$\sigma, \lambda$  constants and  $u$  separable (in space and time)

The forward performance process reduces to a deterministic function

$$U(x, t) = u(x, t)$$

$$u(x, t) = -e^{-x + \frac{t}{2}} \quad \text{or} \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)} \lambda^2 t}$$

Horizon-unbiased utilities; Henderson-Hobson (2006)

- $a(x, t) = k_t$ ,  $k_t \in \mathcal{F}_t$

$$U(x, t) = u(x, A_t) + k_t \cdot W_t$$

- Choulli et al. (2006)

## The “market-view” case

$$a = U\phi, \quad \phi \text{ is a } d\text{-dim } \mathcal{F}_t\text{-adapted process}$$

- The forward performance SPDE becomes

$$dU = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}U (\lambda + \phi)|^2}{\mathcal{A}^2 U} dt + U\phi \cdot dW$$

- Define the processes  $Z$  and  $A$  by

$$dZ = Z\phi \cdot dW \quad \text{and} \quad Z_0 = 1$$

and

$$A_t = \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds$$

- The process  $U = U(x, t)$

$$U(x, t) = u(x, A_t) Z_t$$

with  $u$  solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution

## The “benchmark” case

$a(x, t) = -xU_x(x, t)\delta$ ,  $\delta$  is a  $d$ -dim  $\mathcal{F}_t$ -adapted process

- The forward performance SPDE becomes

$$dU(x, t) = \frac{1}{2} \frac{\left| \sigma_t \sigma_t^+ (U_x(x, t) (\lambda_t - \delta_t) - xU_{xx}(x, t)) \right|^2}{U_{xx}(x, t)} dt - xU_x(x, t) \delta_t \cdot dW_t$$

- Define the processes  $Y$  and  $A$  by

$$dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1$$

and

$$A_t = \int_0^t \left| \sigma_s \sigma_s^+ \lambda_s - \delta_s \right|^2 ds.$$

- Assume  $\sigma \sigma^+ \delta = \delta$
- The process

$$U = U(x, t) = u\left(\frac{x}{Y_t}, A_t\right)$$

with  $u$  as before is a forward performance.

## A more general case

$$a(x, t) = -xU_x(x, t)\delta + U(x, t)\phi$$

- Recall the "benchmark" and "market view processes"

$$dY_t = Y_t\delta_t(\lambda_t dt + dW_t) \quad \text{with} \quad Y = 1$$

and

$$dZ_t = Z_t\phi_t \cdot dW_t \quad \text{with} \quad Z = 1$$

- Define the process

$$A_t = \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s) - \delta_s|^2 ds$$

- The process

$$U = U(x, t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

is a forward performance

MZ (2006, 2007)

## The u-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \quad t > 0,$$

with  $u_0(x) = U(x, 0)$ .

## The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = -\frac{u_x}{u_{xx}}$$

## Modelling considerations

## Three related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2}r^2 r_{xx} = 0 \\ r(x, 0) = r_0(x) \end{cases} \quad (\text{FDE})$$

Conductivity :  $r^2$

- The transport equation

$$u_t + \frac{1}{2}ru_x = 0$$

with  $u_0$  such that  $r_0 = r(x, 0) = -\frac{u_0'(x)}{u_0''(x)}$

- Porous medium equation for risk aversion  $\gamma = r^{-1}$

$$\gamma_t = \frac{1}{2}F(\gamma)_{xx} \quad \text{with} \quad F(\gamma) = \gamma^{-1}$$

## An example of local risk tolerance

(MZ (2006) and Z-Zhou (2007))

$$r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0$$

### (Very) special cases

$$r(x, t; 0, \beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in R$$

$$r(x, t; 1, 0) = |x| \quad \longrightarrow \quad u(x, t) = \log x - \frac{t}{2}, \quad x > 0$$

$$r(x, t; \alpha, 0) = \sqrt{\alpha} |x| \quad \longrightarrow \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)} t}, \quad x \geq 0, \quad \gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

## Optimal allocations



## Optimal allocations

- Let  $X_t^*$  be the optimal wealth,  $Y_t$  the benchmark and  $A_t$  the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\tilde{X}_t^* \triangleq \frac{X_t^*}{Y_t} \quad \text{and} \quad \tilde{R}_t^* \triangleq r(\tilde{X}_t^*, A_t)$$

### Optimal (benchmarked) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^*$$

$$m_t = \sigma_t^+ \delta_t \quad n_t = \sigma_t^+ (\lambda_t + \phi_t - \delta_t)$$

## A system of SDEs at the optimum

$$\widetilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widetilde{R}_t^* = r(\widetilde{X}_t^*, A_t)$$

$$\begin{cases} d\widetilde{X}_t^* = r(\widetilde{X}_t^*, A_t)(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\ d\widetilde{R}_t^* = r_x(\widetilde{X}_t^*, A_t) d\widetilde{X}_t^* \end{cases}$$

Key role in proving the above plays the fact that the local risk tolerance solves  
the fast-diffusion equation

**The optimal wealth and portfolios are explicitly constructed  
if the function  $r(x, t)$  is known**

## Optimal processes and harmonic functions



## Complete construction

### Utility inputs and harmonic functions

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}} \iff h_t + \frac{1}{2} h_{xx} = 0$$

### Harmonic functions and positive Borel measures

$$h(x, t) \iff \nu(dy)$$

### Optimal wealth process

$$X^* = h \left( h^{(-1)}(x, 0) + A + M, A \right) \quad M = \int_0^t \lambda \cdot dW_s, \quad \langle M \rangle = A$$

### Optimal portfolio process

$$\pi^* = h_x \left( h^{(-1)}(X^*, A), A \right) \sigma^+ \lambda$$

### The measure $\nu$ emerges as the defining element

$$\nu \Rightarrow h \Rightarrow u$$

How do we choose  $\nu$  and what does it represent for the investor's risk attitude?

## Concave utility inputs and increasing harmonic functions

- Increasing harmonic function  $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$  is represented as

$$h(x, t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2t} - 1}{y} \nu(dy)$$

- The associated utility input  $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$  is then given by the concave function

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x, s) + \frac{s}{2} h_x(h^{(-1)}(x, s), s)} ds + \int_0^x e^{-h^{(-1)}(z, 0)} dz$$

**The support of the measure  $\nu$  plays a key role in the form of the range of  $h$  and, as a result, in the form of the domain and range of  $u$  as well as in its asymptotic behavior (Inada conditions)**

## Increasing harmonic functions and local risk tolerance

- If  $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$  is an increasing harmonic function then  $r : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}^+$

$$r(x, t) = h_x \left( h^{(-1)}(x, t), t \right) = \int_{\mathbb{R}} e^{yh^{(-1)}(x, t) - \frac{1}{2}y^2t} \nu(dy)$$

is a local risk tolerance function solving the FDE

- Recall that the optimal wealth and portfolio processes solve  $\left( A = \int_0^t |\sigma \sigma^+ \lambda|^2 ds \right)$

$$dX_t^* = r(X_t^*, A_t) \sigma_t \sigma_t^+ \lambda_t (\lambda_t dt + dW_t)$$

$$d\pi_t^* = r_x(X_t^*, A_t) dX_t^*$$

- One then deduces the explicit formulae  $\left( M = \int_0^t \sigma \sigma^+ \lambda \cdot dW \right)$

$$X^* = h \left( h^{(-1)}(x, 0) + A + M, A \right) \quad \text{and} \quad \pi^* = h_x \left( h^{(-1)}(X^*, A), A \right) \sigma^+ \lambda$$

## Examples

- $\nu(dy) = \delta_0$ , where  $\delta_0$  is a Dirac measure at 0. Then,

$$h(x, t) = x \quad \text{and} \quad r(x, t) = 1,$$

and

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-x+\frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x+\frac{t}{2}}$$

- $\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a})$ ,  $a, b > 0$ , where  $\delta_{\pm a}$  is a Dirac measure at  $\pm a$ . Then,

$$h(x, t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax) \quad \text{and} \quad r(x, t) = \sqrt{a^2 x^2 + b^2 e^{-a^2 t}}$$

If,  $\alpha = 1$ , then

$$u(x, t) = \frac{1}{2} \left( \ln \left( x + \sqrt{x^2 + b^2 e^{-t}} \right) - \frac{e^t}{b^2} x \left( x - \sqrt{x^2 + b^2 e^{-t}} \right) - \frac{t}{2} \right)$$

Explicit solutions are obtained for  $\alpha \neq 1$  as well (Z-Zhou (2007))

## Inferring investor's preferences



# Distributional properties of the optimal wealth process

## The case of deterministic market price of risk

Using the explicit representation of  $X^{*,x}$  we can compute the distribution, density, quantile and moments of the optimal wealth process.

- $$\mathbb{P} \left( X_t^{*,x} \leq y \right) = N \left( \frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}} \right)$$
- $$f_{X_t^{*,x}}(y) = n \left( \frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}} \right) \frac{1}{r(y, A_t)}$$
- $$y_p = h \left( h^{(-1)}(x, 0) + A_t + \sqrt{A_t} N^{(-1)}(p), A_t \right)$$
- $$EX_t^{*,x} = h \left( h^{(-1)}(x, 0) + A_t, 0 \right)$$

# Inferring investor's preferences from desirable distributional properties of her optimal wealth

Generalizing Sharpe's (2007) single-period approach

## Investor's investment targets

- Example 1:  $x \rightarrow E \left( X_T^{*,x} \right)$  linear  
 $\Rightarrow \nu = \delta_\gamma \Rightarrow$  power preferences
- Example 2:  $m(t) = E \left( X_t^{*,x_0} \right)$   
 $\Rightarrow m \left( A_t^{(-1)} \right) = \int_0^\infty e^{yt} \nu(dy) \Rightarrow \nu$
- Example 3: target a wealth distribution  $\Rightarrow$  attainable distributions?

**Distributional properties of optimal wealth reveal - via the underlying measure - information about the investor's preferences**

## Summary

- Stochastic evolution of performance criteria
- Traditional and alternative formulations
- The concept of performance volatility process
- SPDE for the value function/performance processes
- Explicit construction of optimal wealth and portfolio processes
- Inference of investor's preferences from here "wish" list

**The case**  $a(x, t) = H(U(x, t))$



## Example

- Risk neutrality ( $\lambda \equiv 0$ ) ;  $\sigma_t = 1, \quad d = k = 1$

$$dU = \frac{1}{2} \frac{a_x^2(x, t)}{U_{xx}} dt + a(x, t) dW$$

$$a \stackrel{=}{=} H(U) \quad \frac{1}{2} \frac{U_x^2}{U_{xx}} (H'(U))^2 dt + H(U) dW$$

- Let  $G : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be such that

$$G_y(z, y) = H(G(z, y))$$

$$G(z, 0) = z$$

- Solution  $U(x, t) = G(v(x, t), W_t)$

$v(x, t)$  is a finite variation process solving

$$dv(x, t) = F(x, v, v_x, v_{xx}, H(v), H'(v), W) dt$$

- Optimal portfolio

$$\pi_t^* = - \frac{(H(G(v(X_t^{\pi,*}, t), W)))_x}{(G(v(X_t^{\pi,*}, t), W))_{xx}}$$

- Optimal wealth

$$X_t^{\pi,*} = x - \int_0^t \frac{(H(G(v(X_s^{\pi,*}, s), W_s)))_x}{(G(v(X_s^{\pi,*}, s), W_s))_{xx}} dW_s$$

**The case**  $a(x, t) = H(U_x(x, t))$



## Example

- Risk neutrality ( $\lambda \equiv 0$ ) ;  $\sigma_t = 1, d = k = 1$

$$dU = \frac{1}{2} \frac{a_x^2(x, t)}{U_{xx}} dt + a dW$$

$$a = \underline{H(U_x)} \quad \frac{1}{2} \left( H'(U_x) \right)^2 U_{xx} dt + H(U_x) dW$$

- Solution

$$U(x, t) = u(x, W_t)$$

$$u_t = H(u_x) ; \quad u(x, 0) = u_0(x)$$

- Choice of Hamiltonians?

- Optimal policy

$$\pi_t^* = -H'(u_x(X_t^{\pi^*}, W_t))$$

- Optimal wealth process

$$X_t^* = x - \int_0^t H'(u_x(X_s^*, W_s)) dW_s$$

- Special case:  $a(x, t) = U_x(x, t)$

$$dU = \frac{1}{2} U_{xx} dt + U_x dW$$

$$u_t = u_x ; \quad u(x, 0) = u_0(x)$$

$$U(x, t) = u(x, W_t) = u_0(x + W_t)$$

Optimal policy:  $\pi_t^* = -1, \quad t > 0$

Optimal wealth:  $X_t^* = x - W_t, \quad t > 0$