SPDE and portfolio choice

(joint work with M. Musiela)

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Performance measurement of investment strategies

Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \le i \le k \qquad \begin{cases} dS_t^i = S_t^i\Big(\mu_t^i dt + \sigma_t^i \cdot dW_t\Big)\ , & S_0^i > 0 \\ dB_t = r_t B_t dt\ , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of an $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t\,\mathbb{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Market environment

- Self-financing investment strategies π_t^0, $\pi_t = (\pi_t^1, \ldots, \pi_t^i, \ldots, \pi_t^k)$

- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t \, dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)$$

Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

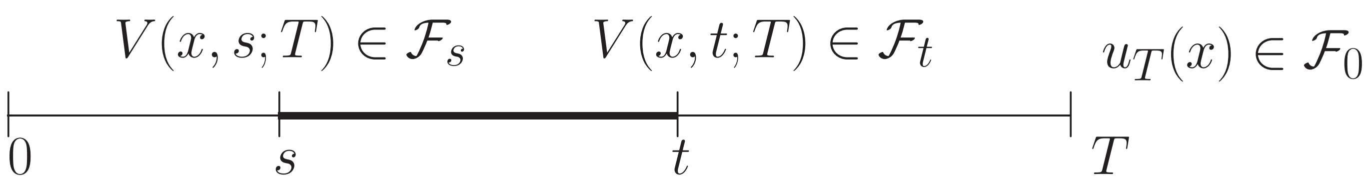
$$V(x,s;T) = \sup_\pi E_\mathbb{P}(u_T(X_T^\pi)|\mathcal{F}_s; X_s^\pi = x)$$

$$V(x,s;T) = \sup_\pi E_\mathbb{P}(V(X_t^\pi,t;T)|\mathcal{F}_s; X_s^\pi = x) \qquad \text{(DPP)}$$

$$V(x,s;T) = E_\mathbb{P}(V(X_t^{\pi^*},t;T)|\mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility for all $t \in [0,T)$

The value function process



- For each self-financing strategy, represented by π, the associated wealth X_t^{π} satisfies

$$E_{\mathbb{P}}(V(X_t^{\pi}, t; T)|\mathcal{F}_s) \leq V(X_s^{\pi}, s; T) \ , \qquad 0 \leq s \leq t \leq T$$

- There exists a self-financing strategy, represented by π^*, for which the associated wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(V(X_t^{\pi^*}, t; T)|\mathcal{F}_s) = V(X_s^{\pi^*}, s; T) \ , \qquad 0 \leq s \leq t \leq T$$

- At expiration, $\quad V(x, T; T) = u_T(x) \in \mathcal{F}_0$

Study of the value function process

- "Arbitrary" environments

  Duality methods

  Martingale representation results

- Markovian environments

  HJB equation

  Feedback optimal controls

  Weak solutions

  ⋮

A stochastic PDE for the value function process

Intuition

- Assume that, for $t \in [0, T]$, the value function $V(x,t)$ solves

$$dV\left(x,t\right) = b\left(x,t\right)dt + a\left(x,t\right) \cdot dW_t$$

where b, a are $\mathcal{F}_t-$measurable processes.

- Recall that for an arbitrary admissible portfolio π, the associated wealth process, X^{π}, solves

$$dX_t^{\pi} = \sigma_t \pi_t \left(\lambda_t dt + dW_t\right)$$

- Applying the Ito-Ventzell formula to $V\left(X_t^{\pi}, t\right)$ yields

$$dV\left(X_t^{\pi}, t\right) = b\left(X_t^{\pi}, t\right)dt + a\left(X_t^{\pi}, t\right) \cdot dW_t$$

$$+V_x\left(X_t^{\pi}, t\right)dX_t^{\pi} + \frac{1}{2}V_{xx}\left(X_t^{\pi}, t\right)d\left\langle X^{\pi}\right\rangle_t + a_x\left(X_t^{\pi}, t\right) \cdot d\left\langle W, X^{\pi}\right\rangle_t$$

$$= \left(b\left(X_t^{\pi}, t\right) + V_x\left(X_t^{\pi}, t\right)\sigma_t\pi_t \cdot \lambda_t + \sigma_t\pi_t \cdot a_x(X_t^{\pi}, t) + \frac{1}{2}V_{xx}\left(X_t^{\pi}, t\right)\left|\sigma_t\pi_t\right|^2\right)dt$$

$$+\left(a\left(X_t^{\pi}, t\right) + V_x\left(X_t^{\pi}, t\right)\sigma_t\pi_t\right) \cdot dW_t$$

Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity

$$\sup_{\pi} \left(V_x \left(X_t^{\pi}, t \right) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_t^{\pi}, t) + \frac{1}{2} V_{xx} \left(X_t^{\pi}, t \right) \left| \sigma_t \pi_t \right|^2 \right)$$

  is well defined

- Calculating the optimum π^* yields

$$\pi_t^* = -\sigma_t^+ \frac{V_x \left(X_t^{\pi^*}, t \right) \lambda_t + a_x \left(X_t^{\pi^*}, t \right)}{V_{xx} \left(X_t^{\pi^*}, t \right)}$$

- Deduce that the above supremum is given by

$$M^* \left(X_t^{\pi^*}, t \right) = - \frac{\left| \sigma_t \sigma_t^+ \left(V_x \left(X_t^{\pi^*}, t \right) \lambda_t + a_x \left(X_t^{\pi^*}, t \right) \right) \right|^2}{2 V_{xx} \left(X_t^{\pi^*}, t \right)}$$

- The drift coefficient b must satisfy

$$b \left(X_t^{\pi^*}, t \right) = -M^* \left(X_t^{\pi^*}, t \right)$$

SPDE for the value function process

- Market $(\sigma_t, \sigma_t^+, \lambda_t)$; volatility $a(x,t) \in \mathcal{F}_t$

$$dV = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}(V\lambda + a)|^2}{\mathcal{A}^2 V} dt + a \cdot dW$$

$$V(x,T) = u_T(x) \in \mathcal{F}_0 \; ; \qquad \mathcal{A} = \frac{\partial}{\partial x}$$

- Feedback optimal portfolio vector

$$\pi_t^* = \pi^*(X_t^{\pi,*}, t) = -\sigma^+ \frac{\mathcal{A}(V\lambda + a)}{\mathcal{A}^2 V}(X_t^{\pi,*} t)$$

- Choices for the volatility process a?

A Markovian example

- $r_t = r(Y_t), \quad \mu_t = \mu(Y_t), \quad \sigma_t = \sigma(Y_t)$

  $dY_t = \theta(Y_t)\,dt + \Theta^T(Y_t)\,dW_t$

- Value function

$$v(x,y,t;T) = \sup_{\pi} E\left(u_T(X_T^{\pi}) \mid X_t^{\pi} = x,\ Y_t = y\right)$$

- HJB equation

$$v_t + \sup_{\pi}\left(\frac{1}{2}|\sigma\pi|^2 v_{xx} + \sigma\pi\cdot\sigma\sigma^{+}(\lambda v_x + \Theta v_{xy}) + \frac{1}{2}\Theta^T\Theta\cdot v_{yy} + \theta\cdot v_y\right)$$

$$= v_t - \frac{1}{2}\,\frac{|\sigma\sigma^{+}(v_x\lambda + \Theta v_{xy})|^2}{v_{xx}} + \frac{1}{2}\Theta^T\Theta\cdot v_{yy} + \theta\cdot v_y = 0$$

- The SPDE for the value function process

$$V(x,t) = v(x, Y_t, t; T)$$

$$dV(x,t) = v_t\, dt + v_y \cdot dY + \frac{1}{2} v_{yy} \cdot d\langle Y \rangle$$

$$\overset{\text{HJB}}{=} \left(\frac{1}{2} \frac{|\sigma\sigma^+(v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} - \frac{1}{2} \Theta^T \Theta \cdot v_{yy} - \theta \cdot v_y \right) dt$$

$$+ v_y \cdot \left(\theta\, dt + \Theta^T dW \right) + \frac{1}{2} v_{yy} \cdot \Theta^T \Theta\, dt$$

$$= \frac{1}{2} \frac{|\sigma\sigma^+(V_x \lambda + a_x(x,t))|^2}{V_{xx}} dt + a(x,t) \cdot dW$$

- The volatility process is uniquely determined: $a(x,t) = \Theta v_y(x, Y_t, t; T)$

Going beyond the deterministic terminal utility problem

Motivation (partial)

- Terminal utility might be ω-dependent

  Liability management, indifference valuation

  $$u_T(x,\omega) = -e^{-\gamma(x - C_T(\omega))}\ ; \qquad C_T \in \mathcal{F}_T$$

  Numeraire consistency

  $$u_T(x,\omega) = -e^{-\gamma_T(\omega)x}$$

- Need to extend the value function process beyond T

- Need to manage liabilities of arbitrary maturities

How do we formulate investment performance criteria?

Investment performance process

$U(x,t)$ is an $\mathcal{F}_t$-adapted process, $\qquad t \geq 0$

- The mapping $x \rightarrow U(x,t)$ is increasing and concave

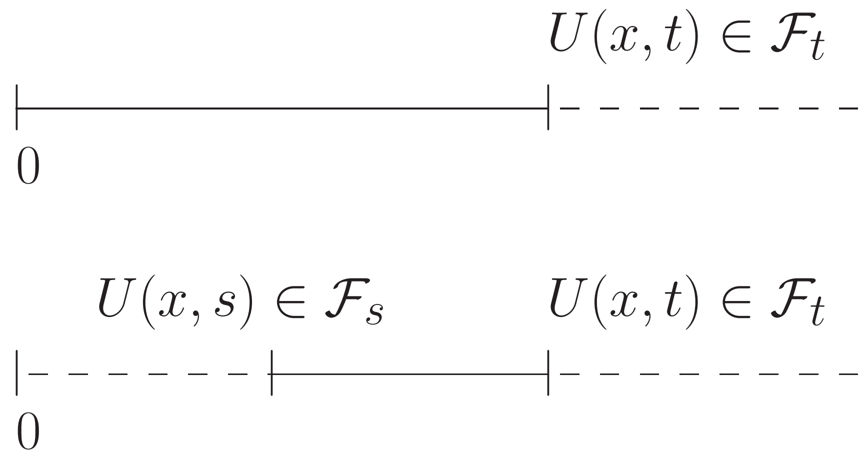
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t^π satisfies

$$E_{\mathbb{P}}(U(X_t^\pi, t) \mid \mathcal{F}_s) \leq U(X_s^\pi, s), \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

Optimality across times



$$U(x,s) = \sup_{\mathcal{A}} E(U(X_t^{\pi}, t) | \mathcal{F}_s,\ X_s = x)$$

- What is the meaning of this process?
- Does such a process aways exist?
- Is it unique?

Forward performance process

A datum $u_0(x)$ is assigned at the beginning of

the trading horizon, $t = 0$

$$U(x, 0) = u_0(x)$$

Forward in time criteria

$$E_{\mathbb{P}}(U(X_t^{\pi}, t)|\mathcal{F}_s) \leq U(X_s^{\pi}, s), \qquad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t)|\mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

Many difficulties due to "inverse in time"

nature of the problem

The forward performance SPDE

The forward performance SPDE

Let $U(x,t)$ be an $\mathcal{F}_t-$measurable process such that the mapping $x \rightarrow U(x,t)$ is increasing and concave. Let also $U = U(x,t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$adapted process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x,t)$ is a forward performance process.

The process a may depend on t, x, U, its spatial derivatives etc.

Optimal portfolios and wealth

At the optimum

- The optimal portfolio vector π^* is given in the feedback form

$$\pi_t^* = \pi^*\left(X_t^*, t\right) = -\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)$$

- The optimal wealth process X^* solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)\left(\lambda dt + dW_t\right)$$

Solutions to the forward performance SPDE

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

Local differential coefficients

$$a\left(x,t\right) = F\left(x,t,U\left(x,t\right),U_{x}\left(x,t\right)\right)$$

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on U_x and U_{xx}
- The equation is not (degenerate) elliptic

Examples

Choices of volatility coefficient

- The zero volatility case: $a(x,t) = 0$

The forward performance SPDE simplifies to

$$dU = \frac{1}{2} \frac{\left| \sigma \sigma^{+} \mathcal{A}(U \lambda) \right|^{2}}{\mathcal{A}^{2} U} dt$$

The process

$$U(x,t) = u(x, A_t) \quad \text{with} \quad A_t = \int_0^t \left| \sigma_s \sigma_s^{+} \lambda_s \right|^2 ds$$

with $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$, increasing and concave with respect to x, and solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)

- $a(x,t)=0$

  σ, λ constants and u separable (in space and time)

The forward performance process reduces to a deterministic function

$$U(x,t)=u(x,t)$$

$$u(x,t)=-e^{-x+\frac{t}{2}} \qquad \text{or} \qquad u(x,t)=\frac{1}{\gamma}x^{\gamma}e^{-\frac{\gamma}{2(1-\gamma)}\lambda^2 t}$$

Horizon-unbiased utilities; Henderson-Hobson (2006)

- $a(x,t)=k_t \ , \quad k_t \in \mathcal{F}_t$

$$U(x,t)=u(x,A_t)+k_t \cdot W_t$$

- Choulli et al. (2006)

The "market-view" case

$a = U\phi, \qquad \phi$ is a $d-$dim $\mathcal{F}_t-$adapted process

- The forward performance SPDE becomes

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}U\left(\lambda+\phi\right)\right|^{2}}{\mathcal{A}^{2}U}dt + U\phi\cdot dW$$

- Define the processes Z and A by

$$dZ = Z\phi\cdot dW \qquad \text{and} \qquad Z_0 = 1$$

and

$$A_t = \int_0^t \left|\sigma_s\sigma_s^{+}(\lambda_s+\phi_s)\right|^{2} ds$$

- The process $U = U\left(x,t\right)$

$$U\left(x,t\right) = u\left(x,A_t\right)Z_t$$

with u solving

$$u_t u_{xx} = \frac{1}{2}u_x^2$$

is a solution

The "benchmark" case

$$a(x,t) = -xU_x(x,t)\,\delta, \quad \delta \text{ is a } d\text{−dim } \mathcal{F}_t\text{−adapted process}$$

- The forward performance SPDE becomes

$$dU(x,t) = \frac{1}{2}\frac{\left|\sigma_t\sigma_t^+\left(U_x(x,t)(\lambda_t-\delta_t) - xU_{xx}(x,t)\right)\right|^2}{U_{xx}(x,t)}dt - xU_x(x,t)\,\delta_t\cdot dW_t$$

- Define the processes Y and A by

$$dY_t = Y_t\delta_t(\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1$$

and

$$A_t = \int_0^t \left|\sigma_s\sigma_s^+\lambda_s - \delta_s\right|^2 ds.$$

- Assume $\sigma\sigma^+\delta = \delta$

- The process

$$U = U(x,t) = u\left(\frac{x}{Y_t}, A_t\right)$$

with u as before is a forward performance.

A more general case

$$a(x,t) = -xU_x(x,t)\,\delta + U(x,t)\,\phi$$

- Recall the "benchmark" and "market view processes"

$$dY_t = Y_t \delta_t \left(\lambda_t dt + dW_t\right) \quad \text{with} \quad Y = 1$$

and

$$dZ_t = Z_t \phi_t \cdot dW_t \quad \text{with} \quad Z = 1$$

- Define the process

$$A_t = \int_0^t \left|\sigma_s \sigma_s^+ \left(\lambda_s + \phi_s\right) - \delta_s\right|^2 ds$$

- The process

$$U = U(x,t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

is a forward performance

MZ (2006, 2007)

The u-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \qquad t > 0,$$

with $u_0(x) = U(x, 0)$.

The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = -\frac{u_x}{u_{xx}}$$

Modelling considerations

Three related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2}r^2 r_{xx} = 0 \\ r(x,0) = r_0(x) \end{cases} \qquad \text{(FDE)}$$

Conductivity : r^2

- The transport equation

$$u_t + \frac{1}{2} r u_x = 0$$

with u_0 such that $r_0 = r\,(x,0) = -\frac{u_0'(x)}{u_0''(x)}$

- Porous medium equation for risk aversion $\gamma = r^{-1}$

$$\gamma_t = \frac{1}{2} F\,(\gamma)_{xx} \qquad \text{with} \qquad F\,(\gamma) = \gamma^{-1}$$

An example of local risk tolerance

(MZ (2006) and Z-Zhou (2007))

$$r(x,t;\alpha,\beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \qquad \alpha, \beta > 0$$

(Very) special cases

$$r(x,t;0,\beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x,t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in R$$

$$r(x,t;1,0) = |x| \quad \longrightarrow \quad u(x,t) = \log x - \frac{t}{2}, \quad x > 0$$

$$r(x,t;\alpha,0) = \sqrt{\alpha}\ |x| \ \longrightarrow \ u(x,t) = \frac{1}{\gamma} x^{\gamma} e^{-\frac{\gamma}{2(1-\gamma)}t}, \ x \geq 0, \ \gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

Optimal allocations

Optimal allocations

- Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$
$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$
$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\widetilde{X}_t^* \triangleq \frac{X_t^*}{Y_t} \qquad \text{and} \qquad \widetilde{R}_t^* \triangleq r(\widetilde{X}_t^*, A_t)$$

Optimal (benchmarked) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = m_t \widetilde{X}_t^* + n_t \widetilde{R}_t^*$$

$$m_t = \sigma_t^+ \delta_t \qquad n_t = \sigma_t^+ (\lambda_t + \phi_t - \delta_t)$$

A system of SDEs at the optimum

$$\widetilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widetilde{R}_t^* = r(\widetilde{X}_t^*, A_t)$$

$$\begin{cases} d\widetilde{X}_t^* = r(\widetilde{X}_t^*, A_t)(\sigma_t \sigma_t^+(\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\,dt + dW_t) \\ d\widetilde{R}_t^* = r_x(\widetilde{X}_t^*, A_t) d\widetilde{X}_t^* \end{cases}$$

Key role in proving the above plays the fact that the local risk tolerance solves the fast-diffusion equation

The optimal wealth and portfolios are explicitly constructed if the function $r(x,t)$ is known

Optimal processes and harmonic functions

Complete construction

Utility inputs and harmonic functions

$$u_t = \frac{1}{2}\,\frac{u_x^2}{u_{xx}} \qquad \Longleftrightarrow \qquad h_t + \frac{1}{2}\,h_{xx} = 0$$

Harmonic functions and positive Borel measures

$$h(x,t) \qquad \Longleftrightarrow \qquad \nu(dy)$$

Optimal wealth process

$$X^* = h\left(h^{(-1)}\,(x,0) + A + M, A\right) \qquad M = \int_0^t \lambda \cdot dW_s, \quad \langle M \rangle = A$$

Optimal portfolio process

$$\pi^* = h_x\left(h^{(-1)}\,(X^*, A)\,, A\right)\sigma^+\lambda$$

The measure ν emerges as the defining element

$$\nu \quad \Rightarrow \quad h \quad \Rightarrow \quad u$$

How do we choose ν and what does it represent for the investor's risk attitude?

Concave utility inputs and increasing harmonic functions

- Increasing harmonic function $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is represented as

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy)$$

- The associated utility input $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is then given by the concave function

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz$$

The support of the measure ν plays a key role in the form of the range of h and, as a result, in the form of the domain and range of u as well as in its asymptotic behavior (Inada conditions)

Increasing harmonic functions and local risk tolerance

- If $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is an increasing harmonic function then $r : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^+$

$$r(x,t) = h_x\left(h^{(-1)}(x,t), t\right) = \int_{\mathbb{R}} e^{yh^{(-1)}(x,t) - \frac{1}{2}y^2 t} \nu(dy)$$

  is a local risk tolerance function solving the FDE

- Recall that the optimal wealth and portfolio processes solve $\left(A = \int_0^t \left|\sigma\sigma^+\lambda\right|^2 ds\right)$

$$dX_t^* = r\left(X_t^*, A_t\right) \sigma_t \sigma_t^+ \lambda_t \left(\lambda_t dt + dW_t\right)$$
$$d\pi_t^* = r_x\left(X_t^*, A_t\right) dX_t^*$$

- One then deduces the explicit formulae $\left(M = \int_0^t \sigma\sigma^+\lambda \cdot dW\right)$

$X^* = h\left(h^{(-1)}(x,0) + A + M, A\right)$ and $\pi^* = h_x\left(h^{(-1)}(X^*, A), A\right)\sigma^+\lambda$

Examples

- $\nu(dy) = \delta_0$, where δ_0 is a Dirac measure at 0. Then,

$$h(x,t) = x \quad \text{and} \quad r(x,t) = 1,$$

and

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-x+\frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x+\frac{t}{2}}$$

- $\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a})$, $a, b > 0$, where $\delta_{\pm a}$ is a Dirac measure at $\pm a$. Then,

$$h(x,t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax) \quad \text{and} \quad r(x,t) = \sqrt{a^2 x^2 + b^2 e^{-a^2 t}}$$

If, $\alpha = 1$, then

$$u(x,t) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{e^t}{b^2} x \left(x - \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{t}{2}\right)$$

Explicit solutions are obtained for $\alpha \neq 1$ as well (Z-Zhou (2007))

Inferring investor's preferences

Distributional properties of the optimal wealth process

The case of deterministic market price of risk

Using the explicit representation of $X^{*,x}$ we can compute the distribution, density, quantile and moments of the optimal wealth process.

- $$\mathbb{P}\left(X_t^{*,x} \leq y\right) = N\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}}\right)$$

- $$f_{X_t^{*,x}}(y) = n\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}}\right)\frac{1}{r(y, A_t)}$$

- $$y_p = h\left(h^{(-1)}(x, 0) + A_t + \sqrt{A_t}N^{(-1)}(p), A_t\right)$$

- $$EX_t^{*,x} = h\left(h^{(-1)}(x, 0) + A_t, 0\right)$$

Inferring investor's preferences from desirable distributional properties of her optimal wealth

Generalizing Sharpe's (2007) single-period approach

Investor's investment targets

- Example 1: $x \rightarrow E\left(X_T^{*,x}\right)$ linear

  $\Rightarrow \quad \nu = \delta_\gamma \quad \Rightarrow$ power preferences

- Example 2: $m(t) = E\left(X_t^{*,x_0}\right)$

  $$\Rightarrow \quad m\left(A_t^{(-1)}\right) = \int_0^\infty e^{yt}\nu(dy) \ \Rightarrow \ \nu$$

- Example 3: target a wealth distribution $\Rightarrow$ attainable distributions?

Distributional properties of optimal wealth reveal - via the underlying measure - information about the investor's preferences

Summary

- Stochastic evolution of performance criteria
- Traditional and alternative formulations
- The concept of performance volatility process
- SPDE for the value function/performance processes
- Explicit construction of optimal wealth and portfolio processes
- Inference of investor's preferences from here "wish" list

The case $a(x,t) = H(U(x,t))$

Example

- Risk neutrality $(\lambda \equiv 0)$; $\sigma_t = 1$, $d = k = 1$

$$dU = \frac{1}{2}\,\frac{a_x^2(x,t)}{U_{xx}}\,dt + a(x,t)\,dW$$

$$\underset{=}{a = H(U)} \quad \frac{1}{2}\,\frac{U_x^2}{U_{xx}}(H'(U))^2\,dt + H(U)\,dW$$

- Let $G : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$G_y(z,y) = H(G(z,y))$$

$$G(z,0) = z$$

- Solution $U(x,t) = G(v(x,t), W_t)$

  $v(x,t)$ is a finite variation process solving

$$dv(x,t) = F(x, v, v_x, v_{xx}, H(v), H'(v), W)\, dt$$

- Optimal portfolio

$$\pi_t^* = -\, \frac{(H(G(v(X_t^{\pi,*}, t), W)))_x}{(G(v(X_t^{\pi,*}, t), W))_{xx}}$$

- Optimal wealth

$$X_t^{\pi,*} = x - \int_0^t \frac{(H(G(v(X_s^{\pi,*}, s), W_s)))_x}{(G(v(X_s^{\pi,*}, s), W_s))_{xx}}\, dW_s$$

The case $a(x,t) = H(U_x(x,t))$

Example

- Risk neutrality $(\lambda \equiv 0)$; $\quad \sigma_t = 1, \quad d = k = 1$

$$dU = \frac{1}{2}\,\frac{a_x^2(x,t)}{U_{xx}}\,dt + a\,dW$$

$$\underset{=}{a = H(U_x)} \quad \frac{1}{2}\left(H'(U_x)\right)^2 U_{xx}\,dt + H(U_x)\,dW$$

- Solution

$$U(x,t) = u(x, W_t)$$

$$u_t = H(u_x) \; ; \quad u(x,0) = u_0(x)$$

- Choice of Hamiltonians?

- Optimal policy

$$\pi_t^* = -H'(u_x(X_t^{\pi,*}, W_t))$$

- Optimal wealth process

$$X_t^* = x - \int_0^t H'(u_x(X_s^*, W_s))\, dW_s$$

- Special case: $a(x,t) = U_x(x,t)$

$$dU = \frac{1}{2} U_{xx}\, dt + U_x\, dW$$

$$u_t = u_x \; ; \quad u(x,0) = u_0(x)$$

$$U(x,t) = u(x, W_t) = u_0(x + W_t)$$

Optimal policy: $\pi_t^* = -1, \qquad t > 0$

Optimal wealth: $X_t^* = x - W_t \; , \qquad t > 0$