SPDE and portfolio choice

(joint work with M. Musiela)

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Performance measurement of investment strategies
Market environment

Riskless and risky securities

• \((\Omega, \mathcal{F}, \mathbb{P})\) ; \(W = (W^1, \ldots, W^d)\)  standard Brownian Motion

• Traded securities

\[
1 \leq i \leq k \quad \begin{cases} 
    dS^i_t = S^i_t \left( \mu^i_t dt + \sigma^i_t \cdot dW_t \right), & S^i_0 > 0 \\
    dB_t = r_t B_t dt, & B_0 = 1
\end{cases}
\]

\(\mu_t, r_t \in \mathbb{R}, \sigma^i_t \in \mathbb{R}^d\)  bounded and \(\mathcal{F}_t\)-measurable stochastic processes

• Postulate existence of an \(\mathcal{F}_t\)-measurable stochastic process \(\lambda_t \in \mathbb{R}^d\) satisfying

\[
\mu_t - r_t 1 = \sigma_t^T \lambda_t
\]

• No assumptions on market completeness
Market environment

- Self-financing investment strategies \( \pi^0, \pi^t = (\pi^1_t, \ldots, \pi^i_t, \ldots, \pi^k_t) \)

- Present value of this allocation

\[
X_t = \sum_{i=0}^{k} \pi^i_t
\]

\[
dX_t = \sum_{i=1}^{k} \pi^i_t \sigma^i_t \cdot (\lambda_t \, dt + dW_t)
\]

\[
= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)
\]
Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

$$V(x, s; T) = \sup_\pi E_{\mathbb{P}}(u_T(X_T^\pi)|\mathcal{F}_s; X_s^\pi = x)$$

$$V(x, s; T) = \sup_\pi E_{\mathbb{P}}(V(X_t^\pi, t; T)|\mathcal{F}_s; X_s^\pi = x) \quad \text{(DPP)}$$

$$V(x, s; T) = E_{\mathbb{P}}(V(X_t^{\pi^*}, t; T)|\mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility for all $t \in [0, T)$
The value function process

\[ V(x, s; T) \in \mathcal{F}_s \quad \text{and} \quad V(x, t; T) \in \mathcal{F}_t \quad \text{and} \quad u_T(x) \in \mathcal{F}_0 \]

- For each self-financing strategy, represented by \( \pi \), the associated wealth \( X_t^\pi \) satisfies

\[ E_\mathbb{P}(V(X_t^\pi, t; T)|\mathcal{F}_s) \leq V(X_s^\pi, s; T) \, , \quad 0 \leq s \leq t \leq T \]

- There exists a self-financing strategy, represented by \( \pi^* \), for which the associated wealth \( X_t^{\pi^*} \) satisfies

\[ E_\mathbb{P}(V(X_t^{\pi^*}, t; T)|\mathcal{F}_s) = V(X_s^{\pi^*}, s; T) \, , \quad 0 \leq s \leq t \leq T \]

- At expiration, \( V(x, T; T) = u_T(x) \in \mathcal{F}_0 \)
Study of the value function process

- “Arbitrary” environments
  - Duality methods
  - Martingale representation results

- Markovian environments
  - HJB equation
  - Feedback optimal controls
  - Weak solutions
A stochastic PDE for the value function process
**Intuition**

- Assume that, for \( t \in [0, T] \), the value function \( V(x, t) \) solves
  \[
  dV(x, t) = b(x, t) \, dt + a(x, t) \cdot dW_t
  \]
  where \( b, a \) are \( \mathcal{F}_t \)-measurable processes.

- Recall that for an arbitrary admissible portfolio \( \pi \), the associated wealth process, \( X^\pi \), solves
  \[
  dX^\pi_t = \sigma_t \pi_t \left( \lambda_t \, dt + dW_t \right)
  \]

- Applying the Itô-Ventzell formula to \( V(X^\pi_t, t) \) yields
  \[
  dV(X^\pi_t, t) = b(X^\pi_t, t) \, dt + a(X^\pi_t, t) \cdot dW_t
  \]
  
  \[
  + V_x(X^\pi_t, t) \, dX^\pi_t + \frac{1}{2} V_{xx}(X^\pi_t, t) \, d\langle X^\pi \rangle_t + a_x(X^\pi_t, t) \cdot d\langle W, X^\pi \rangle_t
  \]
  
  \[
  = \left( b(X^\pi_t, t) + V_x(X^\pi_t, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x(X^\pi_t, t) + \frac{1}{2} V_{xx}(X^\pi_t, t) |\sigma_t \pi_t|^2 \right) \, dt
  \]
  
  \[
  + (a(X^\pi_t, t) + V_x(X^\pi_t, t) \sigma_t \pi_t) \cdot dW_t
  \]
Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity
  \[
  \sup_{\pi} \left( V_x (X_\pi^t, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_\pi^t, t) + \frac{1}{2} V_x (X_\pi^t, t) |\sigma_t \pi_t|^2 \right)
  \]
  is well defined.

- Calculating the optimum \( \pi^* \) yields
  \[
  \pi_t^* = -\sigma_t^+ \frac{V_x (X_t^\pi^*, t) \lambda_t + a_x (X_t^\pi^*, t)}{V_x x (X_t^\pi^*, t)}
  \]

- Deduce that the above supremum is given by
  \[
  M^* (X_t^\pi^*, t) = -\frac{\sigma_t \sigma_t^+ (V_x (X_t^\pi^*, t) \lambda_t + a_x (X_t^\pi^*, t))}{2V_x x (X_t^\pi^*, t)}
  \]

- The drift coefficient \( b \) must satisfy
  \[
  b (X_t^\pi^*, t) = -M^* (X_t^\pi^*, t)
  \]
SPDE for the value function process

- Market \((\sigma_t, \sigma_t^+, \lambda_t)\); volatility \(a(x, t) \in \mathcal{F}_t\)

\[
dV = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(V \lambda + a)|^2}{\mathcal{A}^2 V} \, dt + a \cdot dW
\]

\[
V(x, T) = u_T(x) \in \mathcal{F}_0 ; \quad \mathcal{A} = \frac{\partial}{\partial x}
\]

- Feedback optimal portfolio vector

\[
\pi_t^* = \pi^*(X_t^{\pi^*, t}) = -\sigma^+ \frac{\mathcal{A}(V \lambda + a)}{\mathcal{A}^2 V}(X_t^{\pi^*, t})
\]

- Choices for the volatility process \(a\)?
A Markovian example

- $r_t = r(Y_t)$, $\mu_t = \mu(Y_t)$, $\sigma_t = \sigma(Y_t)$

$$dY_t = \theta(Y_t)\,dt + \Theta^T(Y_t)\,dW_t$$

- Value function

$$v(x, y, t; T) = \sup_\pi E(u_T(X^\pi_T) \mid X^\pi_t = x, Y_t = y)$$

- HJB equation

$$v_t + \sup_\pi \left( \frac{1}{2} \left| \sigma \pi \right|^2 v_{xx} + \sigma \pi \cdot \sigma \sigma^+(\lambda v_x + \Theta v_{xy}) + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y \right)$$

$$= v_t - \frac{1}{2} \frac{|\sigma \sigma^+(v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y = 0$$
• The SPDE for the value function process

\[ V(x, t) = v(x, Y_t, t; T) \]

\[ dV(x, t) = v_t \, dt + v_y \cdot dY + \frac{1}{2} v_{yy} \cdot d\langle Y \rangle \]

HJB \[ \equiv \left( \frac{1}{2} \frac{|\sigma \sigma^+(v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} - \frac{1}{2} \Theta^T \Theta \cdot v_{yy} - \theta \cdot v_y \right) \, dt \]

\[ + v_y \cdot \left( \theta \, dt + \Theta^T \, dW \right) + \frac{1}{2} \, v_{yy} \cdot \Theta^T \Theta \, dt \]

\[ = \frac{1}{2} \frac{|\sigma \sigma^+(V_x \lambda + a_x(x, t))|^2}{V_{xx}} \, dt + a(x, t) \cdot dW \]

• The volatility process is uniquely determined: \[ a(x, t) = \Theta v_y(x, Y_t, t; T) \]
Going beyond the deterministic terminal utility problem
Motivation (partial)

- Terminal utility might be $\omega$-dependent
  
  Liability management, indifference valuation
  
  $$u_T(x, \omega) = -e^{-\gamma(x - C_T(\omega))} ; \quad C_T \in \mathcal{F}_T$$

  Numeraire consistency
  
  $$u_T(x, \omega) = -e^{-\gamma T(\omega)x}$$

- Need to extend the value function process beyond $T$

- Need to manage liabilities of arbitrary maturities

How do we formulate investment performance criteria?
Investment performance process

\[ U(x, t) \] is an \( \mathcal{F}_t \)-adapted process, \hspace{1cm} t \geq 0

- The mapping \( x \to U(x, t) \) is increasing and concave

- For each self-financing strategy, represented by \( \pi \), the associated (discounted) wealth \( X_t^{\pi} \) satisfies

\[
E_P(U(X_t^{\pi}, t) \mid \mathcal{F}_s) \leq U(X_s^{\pi}, s), \quad 0 \leq s \leq t
\]

- There exists a self-financing strategy, represented by \( \pi^* \), for which the associated (discounted) wealth \( X_t^{\pi^*} \) satisfies

\[
E_P(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \quad 0 \leq s \leq t
\]
Optimality across times

\[ U(x, t) \in \mathcal{F}_t \]

\[ U(x, s) \in \mathcal{F}_s \quad U(x, t) \in \mathcal{F}_t \]

\[ U(x, s) = \sup_{\mathcal{A}} E(U(X_t^\pi, t) | \mathcal{F}_s, X_s = x) \]

- What is the meaning of this process?
- Does such a process always exist?
- Is it unique?
Forward performance process

A datum \( u_0(x) \) is assigned at the beginning of the trading horizon, \( t = 0 \)

\[
U(x, 0) = u_0(x)
\]

Forward in time criteria

\[
\begin{align*}
E_{\mathbb{P}}(U(X^\pi_t, t) | \mathcal{F}_s) & \leq U(X^\pi_s, s), \quad 0 \leq s \leq t \\
E_{\mathbb{P}}(U(X^{\pi^*}_t, t) | \mathcal{F}_s) & = U(X^{\pi^*}_s, s), \quad 0 \leq s \leq t
\end{align*}
\]

Many difficulties due to “inverse in time” nature of the problem
The forward performance SPDE
The forward performance SPDE

Let $U(x,t)$ be an $\mathcal{F}_t-$measurable process such that the mapping $x \to U(x,t)$ is increasing and concave. Let also $U = U(x,t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2} \frac{1}{\mathcal{A}^2 U} \left| \sigma \sigma^+ \mathcal{A} (U \lambda + a) \right|^2 dt + a \cdot dW$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$adapted process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x,t)$ is a forward performance process.

The process $a$ may depend on $t, x, U$, its spatial derivatives etc.
Optimal portfolios and wealth

At the optimum

• The optimal portfolio vector $\pi^*$ is given in the feedback form

$$ \pi^*_t = \pi^*(X^*_t, t) = -\sigma \frac{A(U\lambda + a)}{A^2U}(X^*_t, t) $$

• The optimal wealth process $X^*$ solves

$$ dX^*_t = -\sigma^2 + \frac{A(U\lambda + a)}{A^2U}(X^*_t, t)(\lambda dt + dW_t) $$
Solutions to the forward performance SPDE

\[ dU = \frac{1}{2} \frac{\left| \sigma \sigma^+ A (U \lambda + a) \right|^2}{A^2 U} \, dt + a \cdot dW \]

Local differential coefficients

\[ a (x, t) = F (x, t, U (x, t), U_x (x, t)) \]

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on \( U_x \) and \( U_{xx} \)
- The equation is not (degenerate) elliptic
Examples
Choices of volatility coefficient

- The zero volatility case: \( a (x, t) = 0 \)

The forward performance SPDE simplifies to

\[
dU = \frac{1}{2} \frac{\left| \sigma \sigma^+ A (U \lambda) \right|^2}{\mathcal{A}^2 U} \, dt
\]

The process

\[
U (x, t) = u (x, A_t) \quad \text{with} \quad A_t = \int_0^t \left| \sigma_s \sigma^+_s \lambda_s \right|^2 \, ds
\]

with \( u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R} \), increasing and concave with respect to \( x \), and solving

\[
{u_t}u_{xx} = \frac{1}{2} u_x^2
\]

is a solution.

MZ (2006)
Berrier, Rogers and Tehranchi (2007)
• $a(x, t) = 0$

$\sigma, \lambda$ constants and $u$ separable (in space and time)

The forward performance process reduces to a deterministic function

$$U(x, t) = u(x, t)$$

$$u(x, t) = -e^{-\frac{x + \frac{t}{2}}{2}} \quad \text{or} \quad u(x, t) = \frac{1}{\gamma} x \gamma e^{-\frac{\gamma}{2(1-\gamma)} \lambda^2 t}$$

Horizon-unbiased utilities; Henderson-Hobson (2006)

• $a(x, t) = k_t$, $k_t \in F_t$

$$U(x, t) = u(x, A_t) + k_t \cdot W_t$$

• Choulli et al. (2006)
The “market-view” case

\[ a = U \phi, \quad \phi \text{ is a } d-\text{dim } \mathcal{F}_t-\text{adapted process} \]

- The forward performance SPDE becomes

\[
dU = \frac{1}{2} \left| \sigma^{\sigma^+} A U (\lambda + \phi) \right|^2 \frac{dt}{A^2 U} + U \phi \cdot dW
\]

- Define the processes \( Z \) and \( A \) by

\[
dZ = Z \phi \cdot dW \quad \text{and} \quad Z_0 = 1
\]

and

\[
A_t = \int_0^t \left| \sigma_s \sigma_s^+ (\lambda_s + \phi_s) \right|^2 ds
\]

- The process \( U = U (x, t) \)

\[
U (x, t) = u (x, A_t) Z_t
\]

with \( u \) solving

\[
u_t u_{xx} = \frac{1}{2} u_x^2
\]

is a solution
The “benchmark” case

\[ a(x, t) = -xU_x(x, t)\delta, \quad \delta \text{ is a } d-\text{dim } \mathcal{F}_t-\text{adapted process} \]

- The forward performance SPDE becomes

\[
dU(x, t) = \frac{1}{2} \left| \sigma t \sigma^+_t (U_x(x, t)(\lambda_t - \delta_t) - xU_{xx}(x, t)) \right|^2 dt - xU_x(x, t)\delta_t dW_t
\]

- Define the processes \( Y \) and \( A \) by

\[
dY_t = Y_t\delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1
\]

and

\[
A_t = \int_0^t \left| \sigma s \sigma^+_s \lambda_s - \delta_s \right|^2 ds.
\]

- Assume \( \sigma \sigma^+ \delta = \delta \)

- The process

\[
U = U(x, t) = u \left( \frac{x}{Y_t}, A_t \right)
\]

with \( u \) as before is a forward performance.
A more general case

\[ a(x, t) = -xU_x(x, t)\delta + U(x, t)\phi \]

- Recall the "benchmark" and "market view processes"

\[ dY_t = Y_t\delta_t(\lambda_t dt + dW_t) \quad \text{with} \quad Y = 1 \]

and

\[ dZ_t = Z_t\phi_t \cdot dW_t \quad \text{with} \quad Z = 1 \]

- Define the process

\[ A_t = \int_0^t \left| \sigma_s\sigma_s^+ (\lambda_s + \phi_s) - \delta_s \right|^2 ds \]

- The process

\[ U = U(x, t) = u \left( \frac{x}{Y_t}, A_t \right) Z_t \]

is a forward performance

MZ (2006, 2007)
The $u$-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \quad t > 0,$$

with $u_0(x) = U(x, 0)$.

The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = -\frac{u_x}{u_{xx}}$$

Modelling considerations
Three related pdes

• Fast diffusion equation for risk tolerance

\[
\begin{aligned}
rt + \frac{1}{2}r^2 r_{xx} &= 0 \\

\left\{
\begin{array}{l}
\end{array}
\right.
\end{aligned}
\]

(FDE)

\[r(x, 0) = r_0(x)\]

Conductivity : \(r^2\)

• The transport equation

\[
\begin{aligned}
{u}_t + \frac{1}{2}r u_x &= 0 \\

\text{with } u_0 \text{ such that } r_0 &= r(x, 0) = -\frac{u_0'(x)}{u_0''(x)}
\end{aligned}
\]

• Porous medium equation for risk aversion \(\gamma = r^{-1}\)

\[
\gamma_t = \frac{1}{2}F(\gamma)_{xx} \quad \text{with} \quad F(\gamma) = \gamma^{-1}
\]
An example of local risk tolerance

(MZ (2006) and Z-Zhou (2007))

\[ r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0 \]

(Very) special cases

\[ r(x, t; 0, \beta) = \sqrt{\beta} \quad \rightarrow \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in R \]

\[ r(x, t; 1, 0) = |x| \quad \rightarrow \quad u(x, t) = \log x - \frac{t}{2}, \quad x > 0 \]

\[ r(x, t; \alpha, 0) = \sqrt{\alpha} \ |x| \quad \rightarrow \quad u(x, t) = \frac{1}{\gamma} x \gamma e^{-\frac{\gamma}{2(1-\gamma)} t}, \quad x \geq 0, \quad \gamma = \frac{\sqrt{\alpha - 1}}{\sqrt{\alpha}} \]
Optimal allocations
Optimal allocations

• Let $X_t^*$ be the optimal wealth, $Y_t$ the benchmark and $A_t$ the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

• Define

$$\tilde{X}_t^* \triangleq \frac{X_t^*}{Y_t} \quad \text{and} \quad \tilde{R}_t^* \triangleq r(\tilde{X}_t^*, A_t)$$

Optimal (benchmark) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^*$$

$$m_t = \sigma_t^+ \delta_t \quad n_t = \sigma_t^+ (\lambda_t + \phi_t - \delta_t)$$
A system of SDEs at the optimum

\[ \tilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \tilde{R}_t^* = r(\tilde{X}_t^*, A_t) \]

\[
\begin{aligned}
&d\tilde{X}_t^* = r(\tilde{X}_t^*, A_t)(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\
&d\tilde{R}_t^* = r_x(\tilde{X}_t^*, A_t) d\tilde{X}_t^*
\end{aligned}
\]

Key role in proving the above plays the fact that the local risk tolerance solves the fast-diffusion equation

The optimal wealth and portfolios are explicitly constructed if the function \( r(x, t) \) is known
Optimal processes and harmonic functions
Complete construction

Utility inputs and harmonic functions

\[ u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}} \quad \iff \quad h_t + \frac{1}{2} h_{xx} = 0 \]

Harmonic functions and positive Borel measures

\[ h(x, t) \quad \iff \quad \nu(dy) \]

Optimal wealth process

\[ X^* = h \left( h^{-1}(x, 0) + A + M, A \right) \quad M = \int_0^t \lambda \cdot dW_s, \quad \langle M \rangle = A \]

Optimal portfolio process

\[ \pi^* = h_x \left( h^{-1}(X^*, A), A \right) \sigma^+ \lambda \]

The measure \( \nu \) emerges as the defining element

\[ \nu \quad \Rightarrow \quad h \quad \Rightarrow \quad u \]

How do we choose \( \nu \) and what does it represent for the investor’s risk attitude?
Concave utility inputs and increasing harmonic functions

- Increasing harmonic function $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ is represented as
  \[
  h(x, t) = \int_{\mathbb{R}} \frac{e^{y(x - \frac{1}{2}y^2t)} - 1}{y} \nu(dy)
  \]

- The associated utility input $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ is then given by the concave function
  \[
  u(x, t) = -\frac{1}{2} \int_{0}^{t} e^{-h(-1)(x, s) + \frac{s}{2}h_x(h(-1)(x, s), s)} ds + \int_{0}^{x} e^{-h(-1)(z, 0)} dz
  \]

The support of the measure $\nu$ plays a key role in the form of the range of $h$ and, as a result, in the form of the domain and range of $u$ as well as in its asymptotic behavior (Inada conditions)
Increasing harmonic functions and local risk tolerance

- If $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is an increasing harmonic function then $r : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^+$

$$r(x, t) = h_x \left(h^{(-1)}(x, t), t\right) = \int_{\mathbb{R}} e^{y h(-1)(x, t)} - \frac{1}{2} y^2 t \nu(dy)$$

is a local risk tolerance function solving the FDE

- Recall that the optimal wealth and portfolio processes solve

$$(A = \int_0^t \left| \sigma \sigma^+ \lambda \right|^2 ds)$$

$$dX^*_t = r(X^*_t, A_t) \sigma^+_t \lambda_t (\lambda_t dt + dW_t)$$

$$d\pi^*_t = r_x (X^*_t, A_t) dX^*_t$$

- One then deduces the explicit formulae

$$(M = \int_0^t \sigma \sigma^+ \lambda \cdot dW)$$

$$X^* = h \left(h^{(-1)}(x, 0) + A + M, A\right) \quad \text{and} \quad \pi^* = h_x \left(h^{(-1)}(X^*, A), A\right) \sigma^+ \lambda$$
Examples

• $\nu(dy) = \delta_0$, where $\delta_0$ is a Dirac measure at 0. Then,

$$h(x, t) = x \quad \text{and} \quad r(x, t) = 1,$$

and

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-x + \frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x + \frac{t}{2}}$$

• $\nu(dy) = \frac{b}{2} (\delta_a + \delta_{-a})$, $a, b > 0$, where $\delta_{\pm a}$ is a Dirac measure at $\pm a$. Then,

$$h(x, t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax) \quad \text{and} \quad r(x, t) = \sqrt{a^2 x^2 + b^2 e^{-a^2 t}}$$

If, $\alpha = 1$, then

$$u(x, t) = \frac{1}{2} \left( \ln \left( x + \sqrt{x^2 + b^2 e^{-t}} \right) - \frac{e^t}{b^2 x} \left( x - \sqrt{x^2 + b^2 e^{-t}} \right) - \frac{t}{2} \right)$$

Explicit solutions are obtained for $\alpha \neq 1$ as well (Z-Zhou (2007))
Inferring investor’s preferences
Distributional properties of the optimal wealth process

The case of deterministic market price of risk

Using the explicit representation of $X^{*,x}$ we can compute the distribution, density, quantile and moments of the optimal wealth process.

- $\mathbb{P}(X^{*,x}_t \leq y) = N \left( \frac{h^{-1}(y, A_t) - h^{-1}(x, 0) - A_t}{\sqrt{A_t}} \right)$

- $f_{X^{*,x}_t}(y) = n \left( \frac{h^{-1}(y, A_t) - h^{-1}(x, 0) - A_t}{\sqrt{A_t}} \right) \frac{1}{r(y, A_t)}$

- $y_p = h \left( h^{-1}(x, 0) + A_t + \sqrt{A_t N^{-1}(p), A_t} \right)$

- $EX^{*,x}_t = h \left( h^{-1}(x, 0) + A_t, 0 \right)$
Inferring investor’s preferences from desirable distributional properties of her optimal wealth

Generalizing Sharpe’s (2007) single-period approach

Investor’s investment targets

- Example 1: \( x \rightarrow E(X^*_T,x) \) linear
  \[ \Rightarrow \quad \nu = \delta \gamma \quad \Rightarrow \quad \text{power preferences} \]

- Example 2: \( m(t) = E(X^*_t,x_0) \)
  \[ \Rightarrow \quad m(A_t^{(-1)}) = \int_0^\infty e^{yt} \nu(dy) \quad \Rightarrow \quad \nu \]

- Example 3: target a wealth distribution \( \Rightarrow \) attainable distributions?

Distributional properties of optimal wealth reveal - via the underlying measure - information about the investor’s preferences
Summary

- Stochastic evolution of performance criteria
- Traditional and alternative formulations
- The concept of performance volatility process
- SPDE for the value function/performance processes
- Explicit construction of optimal wealth and portfolio processes
- Inference of investor’s preferences from here “wish” list
The case \( a(x,t) = H(U(x,t)) \)
Example

- Risk neutrality (λ ≡ 0); σ_t = 1, d = k = 1

\[ dU = \frac{1}{2} \frac{a_x^2(x, t)}{U_{xx}} dt + a(x, t) dW \]

\[ a = H(U) = \frac{1}{2} \frac{U_x^2}{U_{xx}} (H'(U))^2 \] \[ dt + H(U) dW \]

- Let \( G: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be such that

\[ G_y(z, y) = H(G(z, y)) \]

\[ G(z, 0) = z \]
• Solution \[ U(x, t) = G(v(x, t), W_t) \]

\( v(x, t) \) is a finite variation process solving

\[ dv(x, t) = F(x, v, v_x, v_{xx}, H(v), H'(v), W) \, dt \]

• Optimal portfolio

\[ \pi^*_t = - \frac{(H(G(v(X^\pi^*_t, s), W)))_x}{(G(v(X^\pi^*_t, s), W)))_{xx}} \]

• Optimal wealth

\[ X^\pi^*_t = x - \int_0^t \frac{(H(G(v(X^\pi^*_s, s), W_s)))_x}{(G(v(X^\pi^*_s, s), W_s)))_{xx}} \, dW_s \]
The case \( a(x, t) = H(U_x(x, t)) \)
Example

- Risk neutrality \((\lambda \equiv 0)\); \(\sigma_t = 1, \ d = k = 1\)

\[
dU = \frac{1}{2} \frac{a_x^2(x, t)}{U_{xx}} dt + a \, dW
\]

\[
a = H(U_x) \quad \frac{1}{2} \left( H'(U_x) \right)^2 U_{xx} \, dt + H(U_x) \, dW
\]

- Solution

\[
U(x, t) = u(x, W_t)
\]

\[
u_t = H(u_x) \ ; \ u(x, 0) = u_0(x)
\]

- Choice of Hamiltonians?
• Optimal policy

\[ \pi_t^* = -H'(u_x(X_t^{\pi,*}, W_t)) \]

• Optimal wealth process

\[ X_t^* = x - \int_0^t H'(u_x(X_s^*, W_s)) \, dW_s \]
• Special case: \[ a(x, t) = U_x(x, t) \]

\[
dU = \frac{1}{2} U_{xx} \, dt + U_x \, dW
\]

\[ u_t = u_x ; \quad u(x, 0) = u_0(x) \]

\[ U(x, t) = u(x, W_t) = u_0(x + W_t) \]

Optimal policy: \[ \pi_t^* = -1, \quad t > 0 \]

Optimal wealth: \[ X_t^* = x - W_t , \quad t > 0 \]