Innovations and forward utilities

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Topics

• Utility-based measurement of performance

• Utilities and market dynamics

• Optimal allocations and their stochastic evolution

• Efficient frontier
References

Joint work with M. Musiela (BNP Paribas, London)

- “Investments and forward utilities”
  Preprint 2006

- “Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model”
  Indifference Pricing, PUP (2003, 2005)

- “Investment and valuation under backward and forward dynamic utilities in a stochastic factor model”
  to appear in Dilip Madan’s Festschrift (2006)
Utility-based measurement of performance
Dynamic utility measurement

time $t_1$, information $\mathcal{F}_{t_1}$

- asset returns
- constraints
- market view
- away from equilibrium
- benchmark numeraire
- calendar time
- subordination

$MI(t_1) \quad \rightarrow \quad + \quad \downarrow \quad u(x, t_1)$

$U(x, t_1; MI) \in \mathcal{F}_{t_1} \quad \pi(x, t_1; MI) \in \mathcal{F}_{t_1}$
Dynamic utility measurement

time $t_2$, information $\mathcal{F}_{t_2}$

asset returns
constraints
market view
away from equilibrium
benchmark numeraire
calendar time subordination

$MI(t_2) \xrightarrow{-} + \xleftarrow{\downarrow} u(x, t_2)$

$U(x, t_2; MI) \in \mathcal{F}_{t_2}$  $\pi(x, t_2; MI) \in \mathcal{F}_{t_2}$
Dynamic utility measurement

time $t_3$, information $\mathcal{F}_{t_3}$

- asset returns
- constraints
- market view away from equilibrium
- benchmark numeraire
- calendar time subordination

\[ \begin{align*}
M I(t_3) & \quad \longrightarrow \quad + \\
\downarrow & \quad \downarrow \\
U(x, t_3; M I) & \in \mathcal{F}_{t_3} \quad \pi(x, t_3; M I) \in \mathcal{F}_{t_3}
\end{align*} \]
Forward utility measurement

time $t$, information $\mathcal{F}_t$

asset returns
additional
market input

\[ MI(t) \quad \longrightarrow \quad + \quad \leftarrow \quad u(x, t) \]

\[ U(X_t^*, t) \in \mathcal{F}_t \quad \pi^*(X_t^*, t) \in \mathcal{F}_t \]
Forward utility measurement

time $t_1$, information $\mathcal{F}_{t_1}$

asset returns
additional
market input

\[
\begin{align*}
MI(t_1) & \quad --\rightarrow \quad + \quad \leftarrow \quad u(x, t_1) \\
\downarrow & \\
U(\pi^*(X^*_t, t_1), t_1) & \in \mathcal{F}_{t_1}
\end{align*}
\]
Forward utility measurement

time $t_2$, information $\mathcal{F}_{t_2}$

asset returns
additional
market input

\[ MI(t_2) \quad \longrightarrow \quad + \quad \leftarrow \quad u(x, t_2) \]

\[ U(X_{t_2}^*, t_2) \in \mathcal{F}_{t_2} \quad \pi^*(X_{t_2}^*, t_2) \in \mathcal{F}_{t_2} \]
Forward utility measurement

time $t_3$, information $\mathcal{F}_{t_3}$

asset returns
additional
market input

\begin{align*}
MI(t_3) & \quad \rightarrow \quad + \quad \leftarrow \quad u(x, t_3) \\
& \quad \downarrow \\
U(X_{t_3}^*, t_3) & \in \mathcal{F}_{t_3} \quad \pi^*(X_{t_3}^*, t_3) \in \mathcal{F}_{t_3}
\end{align*}
Utility traits

\[ u(x, t) : x \text{ “wealth” and } t \text{ “time”} \]

- Monotonicity \[ u_x(x, t) > 0 \]
- Risk aversion \[ u_{xx}(x, t) < 0 \]
- Impatience \[ u_t(x, t) < 0 \]

Fisher (1913, 1918), Koopmans (1951), Koopmans-Diamond-Williamson (1964) ...
Modelling and management of uncertainty
Investment universe

Riskless and risky assets

• \( (\Omega, \mathcal{F}, \mathbb{P}) \); \( W = (W^1, \ldots, W^d) \) standard Brownian Motion

• Prices

\[
1 \leq i \leq k \quad \begin{cases} 
    dS^i_t = S^i_t(\mu^i_t dt + \sigma^i_t \cdot dW_t) , & S^i_0 > 0 \\
    dB_t = r_t B_t dt , & B_0 = 1
\end{cases}
\]

\( \mu_t, r_t \in \mathbb{R}, \sigma^i_t \in \mathbb{R}^d \) bounded and \( \mathcal{F}_t \)-measurable stochastic processes

• Postulate existence of a \( \mathcal{F}_t \)-measurable stochastic process \( \lambda_t \in \mathbb{R}^d \) satisfying

\[
\mu_t - r_t 1 = \sigma^T_t \lambda_t
\]
Investment universe

- Self-financing investment strategies $\pi_t^0, \pi_t^i, i = 1, \ldots, k$

- Present value of this allocation

\[
X_t = \sum_{i=0}^{k} \pi_t^i
\]

\[
dX_t = \sum_{i=0}^{k} \pi_t^i (\mu_t^i - r_t) \, dt + \sum_{i=0}^{k} \pi_t^i \sigma_t^i \cdot dW_t
\]

\[
= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)
\]

\[
\pi_t = (\pi_t^1, \ldots, \pi_t^k), \quad \mu_t - r_t \, 1 = \sigma_t^T \lambda_t
\]
Forward dynamic utilities
Fundamental characteristics of a forward dynamic utility structure

- Time evolution concurrent with the one of the investment universe
- Consistency with up to date information
- Incorporation of available opportunities and constraints
- Meaningful optimal utility volume
Forward dynamic utility

\[ U(x, t) \] is an \( \mathcal{F}_t \)-adapted process

- As a function of \( x \), \( U \) is increasing and concave
- Utility datum \( U(x, 0) = u_0(x) \) assigned at forward normalization point \( t = 0 \)
- For each self-financing strategy, represented by \( \pi \), the associated (discounted) wealth \( X_t \) satisfies
  \[
  E_P(U(X_{\pi t}^t), t | \mathcal{F}_s) \leq U(X_{\pi s}^s), s \quad 0 \leq s \leq t
  \]
- There exists a self-financing strategy, represented by \( \pi^* \), for which the associated (discounted) wealth \( X_{\pi t}^* \) satisfies
  \[
  E_P(U(X_{\pi t}^*, t | \mathcal{F}_s) = U(X_{\pi s}^*, s) \quad 0 \leq s \leq t
  \]
Construction of forward dynamic utilities
Forward dynamic utilities

Creating the martingale that yields the optimal utility volume

No Markovian assumptions

Stochastic optimization problem "inverse" in time

Key idea

Stochastic input  Variational input
Market    Individual

Maximal utility — Optimal allocation
Forward dynamic utilities

Stochastic input: \((Y_t, Z_t, A_t)\)  
Variational input: \(u(x, t)\)

Benchmark

Time change
\(A_t\)

utility surface
\(u(x, t)\)

Market view
\(Z_t\)

\[U(x, t) = u\left(\frac{x}{Y_t}, A_t\right)Z_t\]
Market input processes
Market input processes

\[(Y_t, Z_t, A_t)\]

These $\mathcal{F}_t$-mble processes do **not** depend on the investor’s variational utility. They reflect and represent, respectively:

- $Y_t$: benchmark
  - numeraire

- $Z_t$: market view away from market equilibrium
  - feasibility and trading constraints

- $A_t$: subordination
Market input processes

- Market environment

\[
dS^i_t = S^i_t \left( \mu^i_t \, dt + \sigma^i_t \cdot dW_t \right)
\]
\[
dB_t = r_t \, B_t \, dt
\]
\[
\mu_t - r_t \, 1 = \sigma_t^T \lambda_t , \quad \mu_t, \sigma_t, r_t \in \mathcal{F}_t
\]

\[\sigma_t^+ : \text{Moore-Penrose matrix inverse}\]

\[\sigma_t^+ = \lim_{\varepsilon \to 0} (\sigma_t^* \sigma_t + \varepsilon I)^{-1} \sigma_t^* \]

\[\sigma_t^+ \text{ : conjugate transpose of } \sigma_t\]

Properties: \[\sigma_t \sigma_t^+ \sigma_t = \sigma_t, \quad \sigma_t^+ \sigma_t \sigma_t^+ = \sigma_t^+\]

\[(\sigma_t \sigma_t^+)^T = \sigma_t \sigma_t^+, \quad (\sigma_t^+ \sigma_t)^T = \sigma_t^+ \sigma_t\]
Market input processes

- Asset price coefficients
  \[ \lambda_t, \sigma_t \in \mathcal{F}_t \]

- Benchmark and/or numeraire

A “replicable” process \( Y_t \) satisfying

\[
\begin{align*}
  dY_t &= Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\
  Y_0 &= 1
\end{align*}
\]

\[ \delta_t \in \mathcal{F}_t, \quad \sigma_t \sigma_t^+ \delta_t = \delta_t \]
Market input processes

• Market views, feasibility and trading constraints

An exponential martingale $Z_t$ satisfying

$$
\begin{cases}
    dZ_t = Z_t \phi_t \cdot dW_t \\
    Z_0 = 1, \quad \phi_t \in \mathcal{F}_t
\end{cases}
$$

• Subordination

A non-decreasing process $A_t$ solving

$$
\begin{cases}
    dA_t = |\delta_t - \sigma_t \sigma_t^+ (\lambda_t + \phi_t)|^2 dt \\
    A_0 = 0
\end{cases}
$$
Variational input – utility surfaces
Utility surface

A model independent variational constraint on
impatience, risk aversion and monotonicity

• Initial utility datum

\[ u_0(x) = u(x, 0) \]

• Fully non-linear pde

\[
\begin{cases}
    u_t \ u_{xx} = \frac{1}{2} u_x^2 \\
    u(x, 0) = u_0(x)
\end{cases}
\]
Utility transport equation

The utility equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

\[ r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)} \]

\[ \begin{cases} \frac{d}{dt} u(t) + \frac{1}{2}r(x, t)u_x = 0 \\ u(x, 0) = u_0(x) \end{cases} \]

Characteristic curves:

\[ \frac{dx(t)}{dt} = \frac{1}{2}r(x(t), t) \]
Construction of utility surface $u(x,t)$ using characteristics

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$

Utility datum $u_0(x)$
Construction of characteristics

\[ \frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t) \]

Utility datum \( u(x, 0) \)

Characteristic curves
Propagation of utility datum along characteristics
Propagation of utility datum along characteristics
Utility surface $u(x, t)$
Two related pdes

- Fast diffusion equation for risk tolerance
  
  \[
  \begin{aligned}
  &r_t + \frac{1}{2}r^2 r_{xx} = 0 \\
  &r(x, 0) = r_0(x)
  \end{aligned}
  \]

  (FDE) Conductivity : \( r^2 \)

- Porous medium equation for risk aversion
  
  \[
  \gamma(x, t) = \frac{1}{r(x, t)}
  \]

  \[
  \begin{aligned}
  &\gamma_t = \left( \frac{1}{\gamma} \right)_{xx} \\
  &\gamma(x, 0) = \frac{1}{r_0(x)}
  \end{aligned}
  \]

  (PME) Pressure : \( r^2 \) and (PME) exponent: \( m = -1 \)
Difficulties

- **Utility equation:** \[ u_t \ u_{xx} = \frac{1}{2} u_x^2 \]
  Inverse problem and fully nonlinear

- **Utility transport equation:** \[ u_t + \frac{1}{2} r(x, t) u_x = 0 \]
  Shocks, solutions past singularities

- **Fast diffusion equation:** \[ r_t + \frac{1}{2} r^2 r_{xx} = 0 \]
  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- **Porous medium equation:** \[ \gamma_t = \left( \frac{1}{\gamma} \right)_{xx} \]
  Majority of results for (PME), \[ \gamma_t = \left( \gamma^m \right)_{xx} \]
  are for \( m > 1 \), partial results for \(-1 < m < 0\)
A rich class of risk tolerance inputs

- Addititively separable risk tolerance

\[
    r^2(x, t; \alpha, \beta) = m(x; \alpha, \beta) + n(t; \alpha, \beta)
\]

Example

\[
    m(x; \alpha, \beta) = \alpha x^2 \quad n(x; \alpha, \beta) = \beta e^{-\alpha t}
\]

\[
    r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0
\]

(Very) special cases

\[
    r(x, t; 0, \beta) = \sqrt{\beta} \quad \rightarrow \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + t}
\]

\[
    r(x, t; 1, 0) = |x| \quad \rightarrow \quad u(x, t) = \log x - t
\]

\[
    r(x, t; \alpha, 0) = \sqrt{\alpha} |x| \quad \rightarrow \quad u(x, t) = \frac{1}{\gamma} x \gamma e^{-2(1-\gamma) t}, \quad \gamma = \frac{\sqrt{\alpha-1}}{\sqrt{\alpha}}
\]
Risk tolerance \[ r(x, t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}} \]
Utility surface $u(x, t)$ generated by

risk tolerance $r(x, t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}}$

Characteristics: \[ \frac{dx(t)}{dt} = \frac{1}{2} \sqrt{0.05x(t)^2 + 15.5e^{-0.05t}} \]
Risk tolerance \( r(x, t) = \sqrt{10x^2 + e^{-10t}} \)
Utility surface $u(x,t)$ generated by

risk tolerance $r(x,t) = \sqrt{10x^2 + e^{-10t}}$

Characteristics: $\frac{dx(t)}{dt} = \frac{1}{2} \sqrt{10x(t)^2 + e^{-10t}}$
Risk tolerance \( r(x, t) = \sqrt{0x^2 + 1} = 1 \)
Utility surface $u(x, t) = -e^{-x + t}$ generated by risk tolerance $r(x, t) = 1$

Characteristics: $\frac{dx(t)}{dt} = \frac{1}{2}$
Risk tolerance \[ r(x, t) = \sqrt{x^2 + 0e^{-t}} = |x| \]
Utility surface $u(x, t) = \log x - t, \ x > 0$ generated by

risk tolerance $r(x) = x$

Characteristics: $\frac{dx(t)}{dt} = \frac{1}{2}x(t)$
Risk tolerance \( r(x, t) = \sqrt{4x^2 + 0e^{-4t}} = 2|x| \)
Utility surface $u(x, t) = 2\sqrt{x} e^{-\frac{t}{2}}, \ x > 0$ generated by

risk tolerance $r(x, t) = 2x$

Characteristics: $\frac{dx(t)}{dt} = x(t)$
Multiplicatively separable risk tolerance

\[ r(x, t; \alpha, \beta) = m(x; \alpha)n(t; \beta) \]

Example

\[ m(x; \alpha) = \varphi(\Phi^{-1}(x; \alpha)) \quad n(t; \beta) = \sqrt{t + \beta}, \quad \beta > 0 \]

\[ \Phi(x; \alpha) = \int_{\alpha}^{x} e^{z^2/2} \, dz \quad \varphi = \Phi' \]

\[ r(x, t; \alpha, \beta) = \varphi(\Phi^{-1}(x; \alpha)) \]

(Very) special cases

\[ m(x; \alpha) = \alpha, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = -e^{-x/\alpha + t} \]

\[ m(x; \alpha) = x, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = \log x - t \]

\[ m(x; \alpha) = \alpha x, \quad n(t; \beta) = 1 \quad \rightarrow \quad u(x, t) = \frac{1}{\gamma} x \gamma e^{-\frac{\gamma}{2(1-\gamma)} t}, \quad \gamma = \frac{\alpha - 1}{\alpha} \]
Risk tolerance \[ r(x, t) = \frac{\varphi(\Phi^{-1}(x; 0.5))}{\sqrt{t + 5}} \]
Utility surface \[ u(x, t) = \Phi(\Phi^{-1}(x; 0.5) - \sqrt{t + 5}) \]

generated by risk tolerance \[ r(x, t) = \frac{\varphi(\Phi^{-1}(x; 0.5))}{\sqrt{t + 5}} \]

Characteristics: \[ \frac{dx(t)}{dt} = \frac{\varphi(\Phi^{-1}(x(t); 0.5))}{\sqrt{t + 5}} \]
Utility function \( u(x, t_0) \)

(fixed time)

\( t_0 = 2 \)
Utility function \( u(x_0, t) \)

(fixed wealth level)

\( x_0 = 3.5 \)
Optimal utility volume
Optimal asset allocation
Optimal utility volume

**Stochastic market input**

\[ \lambda_t, \sigma_t \]

\[ \downarrow \]

benchmark, views subordination

\[ (Y_t, Z_t, A_t) \]

**Variational input**

\[ x, r_0(x) = -\frac{u'_0(x)}{u''_0(x)} \]

\[ \downarrow \]

fast diffusion eqn transport eqn

\[ u(x, t) \]

\[ U(x, t) = u\left(\frac{x}{A_t}, Y_t\right)Z_t \]

Model independent construction!
What is the optimal allocation?

Optimal portfolio processes

\[ \pi_t = (\pi_t^0, \pi_t^1, \ldots, \pi_t^k) \]

can be directly and explicitly characterized

along with the construction of the forward utility!
The structure of optimal portfolios

\[ dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t) \]

**Stochastic input**

- Market
  - \((Y_t, Z_t, A_t)\)
  - \(\lambda_t, \sigma_t, \delta_t, \phi_t\)

**Variational input**

- Individual
  - wealth \(x\)
  - risk tolerance \(r(x, t)\)

\[ \frac{1}{Y_t} \pi_t^* \text{ is a linear combination of (benchmarked) optimal wealth and subordinated (benchmarked) risk tolerance} \]
Optimal asset allocation

- Let $X^*_t$ be the optimal wealth, $Y_t$ the benchmark and $A_t$ the subordination processes

$$dX^*_t = \sigma_t \pi^*_t \cdot (\lambda_t dt + dW_t)$$
$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$
$$dA_t = \left| \sigma_t \sigma^+_t (\lambda_t + \phi_t) - \delta_t \right|^2 dt$$

- Define $r^*_t$ the subordinated (benchmarked) risk tolerance

$$r^*_t = r \left( \frac{X^*_t}{Y_t}, A_t \right)$$

Optimal (benchmarked) portfolios

$$\frac{1}{Y_t} \pi^*_t = \sigma^+_t \left( (\lambda_t + \phi_t)r^*_t + \delta_t \left( \frac{X^*_t}{Y_t} - r^*_t \right) \right)$$
Wealth-risk tolerance stochastic evolution
A system of SDEs at optimum utility volume

\[ \hat{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \hat{r}_t^* = r(\hat{X}_t^*, A_t) \]

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
    d\hat{X}_t^* = \hat{r}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) \, dt + dW_t) \\
    d\hat{r}_t^* = r_x(\hat{X}_t^*, A_t) d\hat{X}_t^*
\end{array}
\right.
\end{aligned}
\]

- Separability of wealth dynamics in terms of risk tolerance and market input
- Sensitivity of risk tolerance in terms of its spatial gradient and changes in optimal wealth
- Utility functional has essentially vanished

Universal representation, no Markovian assumptions
An efficient frontier

Optimal wealth-risk tolerance \((\hat{X}_t^*, \hat{r}_t^*)\) system
of SDEs in original market configuration

\[
\begin{align*}
    d\hat{X}_t^* &= \hat{r}_t^* (\sigma_t \sigma_t^* (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) \, dt + dW_t) \\
    d\hat{r}_t^* &= r_x (\hat{X}_t^*, A_t) \, d\hat{X}_t^*
\end{align*}
\]

- change of measure
  - historical → benchmarked
- change of time
  - Levy’s theorem
An efficient frontier

Optimal wealth-risk tolerance \((x^1_t, x^2_t)\) system of SDEs

in canonical market configuration

\[
\begin{align*}
x^1_t &= \left( \frac{X^*_t}{Y_t} \right) A_t^{(-1)} \\
x^2_t &= r \left( \frac{X^*_t}{Y_t}, A_t \right) A_t^{(-1)}
\end{align*}
\]

\[
\langle M_t \rangle = A_t \quad \quad w_t = M_A^{(-1)}
\]

\[
\begin{align*}
dx^1_t &= x^2_t \, dw_t \\
dx^2_t &= r_x(x^1_t, t) x^2_t \, dw_t
\end{align*}
\]

\[
x^1_0 = \frac{x}{y}, \quad x^2_0 = r_x \left( \frac{x}{y}, 0 \right)
\]
Analytic solution of the efficient frontier SDE system

\[
\begin{align*}
    dx_1^1 &= x_1^2 \, dw_t \\
    dx_1^2 &= r_x(x_1^1, t)x_1^2 \, dw_t
\end{align*}
\]

- Define the budget capacity function \( h(x, t) \) via

\[
x = \int_x^{h(x, t)} \frac{du}{r(u, t)} = \int_x^{h(x, t)} \gamma(u, t) \, du
\]

\( x \): related to symmetry properties of risk tolerance, reflection point of its spatial derivative and risk aversion front
Analytic solutions

The budget capacity function $h$ solves the (inverse) heat equation

$$\begin{cases} 
  h_t + \frac{1}{2}h_{xx} - \frac{1}{2}r_x(x, t)h_x = 0 \\
  h(x, 0) = h_0(x), \quad x = \int_x^{h_0(x)} \frac{du}{r(u, 0)}
\end{cases}$$

Solution of the efficient frontier SDE system

$$\begin{cases} 
  x^1_t = h(z_t, t) \\
  x^2_t = h_z(z_t, t)
\end{cases}$$

$$z_t = h_0^{-1}(x) - \int_0^t \frac{1}{2}r_x(x, s)ds + w_t$$

Using equivalent measure transformations and time change we recover the original pair of optimal (benchmarked) wealth and (benchmark) risk tolerance.
Utility-based performance measurement

Market

Benchmark, views, constraints

Market input processes

Subordination

Investor

Wealth, risk tolerance

Fast diffusion eqn

Transport eqn

Forward evolution

\[ Y_t, Z_t, A_t, x, r(x,t), u(x,t) \]

Optimal utility volume and optimal portfolios

Efficient frontier SDE system

Heat eqn

Fast diffusion eqn

Universal analytic solutions
ありがとうございました

Thank you!