Indifference valuation under forward and backward stochastic risk preferences

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Towards a constitutive analogue of the Black and Scholes theory in incomplete markets
Modelling, optimal behavior, valuation
and risk management

• Market prices of underlying securities

• Risk preference formulation

• Specification of admissible strategies

• Construction of optimal strategies

• Valuation

In complete markets only the first and last step are important
Fundamental elements of an indifference pricing system

Absence of static arbitrage

- Numeraire independence
- Monotonicity, scaling with respect to payoffs
- Monotonicity, robustness with respect to risk aversion
- Translation invariance with respect to replicable risks
- Risk quantification and monitoring
- Zero indifference value for residual risks
Outline

Motivational examples from indifference valuation

- Numeraire invariance
  - Stochastic risk tolerance
  - Stochastic utilities
    - Normalization
    - Backward

- Semigroup property of prices
  - Invariance of optimal utilities across trading horizons
  - Evolution
  - Forward
References

- “Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model”
  Indifference Pricing, PUP (2005, Musiela-Z.)

- “Investment and valuation under backward and forward dynamic utilities in a stochastic factor model”
  Preprint (2005, Musiela-Z.)

- “Numeraire consistency, stochastic risk preferences and indifference valuation”
Pricing blocks in indifference valuation
A toy incomplete model

- Probability space

\[ \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad \mathbb{P}\{\omega_i\} = p_i, \quad i = 1, \ldots, 4 \]

- Two risks

\[
\begin{align*}
S_0 & \quad S^u \quad S^d \\
Y_0 & \quad Y^u \quad Y^d
\end{align*}
\]

- Random variables $S_T$ and $Y_T$

\[
\begin{align*}
S_T(\omega_1) &= S^u, \quad Y_T(\omega_1) = Y^u & S_T(\omega_3) &= S^d, \quad Y_T(\omega_3) = Y^u \\
S_T(\omega_2) &= S^u, \quad Y_T(\omega_2) = Y^d & S_T(\omega_4) &= S^d, \quad Y_T(\omega_4) = Y^d
\end{align*}
\]
Investment opportunities

• We invest the amount $\beta$ in bond ($r = 0$) and the amount $\alpha$ in stock

• Wealth variable

$$X_0 = x, \quad X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0)$$

Indifference price

• For a general claim $C_T$, we define the value function

$$V^{C_T}(x) = \max_{\alpha} E(-e^{-\gamma(X_T-C_T)})$$

• The indifference price is the amount $\nu(C_T)$ for which,

$$V^0(x) = V^{C_T}(x + \nu(C_T))$$
Structural result

Duality techniques yield for general market environments

\[ \nu(C_T) = \sup_Q \left( E_Q(C_T) - \vartheta \right), \]

\[ \vartheta = \frac{1}{\gamma} \left( H(Q|\mathbb{P}) - H(Q|\mathbb{P}) \right) \]

Rouge and El Karoui (2000)  
Frittelli (2000)  
Kabanov and Stricker (2002)  
Delbaen et al. (2000)
Indifference price representation (MZ 2004)

- Arbitrage free prices
  \[ \nu(C_T) = E_{Q^\star}(C_T) \]

  \[ E(\cdot) : \text{linear pricing functional} \]
  \[ Q^\star : \text{the (unique) risk neutral martingale measure} \]

- Indifference prices
  \[ \nu(C_T) = \mathcal{E}_Q(C_T) \]

  \[ \mathcal{E} : \text{pricing functional} \]
  \[ (\text{possibly}) \text{ nonlinear} \]
  \[ \text{payoff independent} \]
  \[ \text{wealth independent} \]
  \[ \text{preference dependent} \]

  \[ Q : \text{pricing measure} \]
  \[ \text{payoff independent} \]
  \[ \text{preference independent} \]
The indifference price

\[ \nu(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q (e^{\gamma C(S_T, Y_T)} \mid S_T) \right) = \mathcal{E}_Q(C_T) \]

\[ Q(Y_T \mid S_T) = P(Y_T \mid S_T) \]
Static arbitrage
Indifference prices in spot and forward units

Spot units

Wealth: \( X^s_T = x + \alpha \left( \frac{S_T}{1+r} - S_0 \right) \)

Value function: \( V^{C,T}(x) = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma \left( X^s_T - \frac{C_T}{1+r} \right)} \right) \)

Pricing condition: \( V^0(x) = V^{s,C,T}(x + \nu^s(C_T)) \)

Pricing measure: \( E_{\mathbb{Q}^s} \left( \frac{S_T}{1+r} \right) = S_0 \) and \( \mathbb{Q}^s(Y_T|S_T) = \mathbb{P}(Y_T|S_T) \)

Indifference price: \( \nu^s(C_T) = E_{\mathbb{Q}^s} \left( \frac{C_T}{1+r} \right) = E_{\mathbb{Q}^s} \left( \frac{1}{\gamma} \log E_{\mathbb{Q}^s} \left( e^{\gamma \frac{C_T}{1+r}} | S_T \right) \right) \)
Forward units

Wealth: \[ X^f_T = X^s_T(1 + r) = f + \alpha(F_T - F_0) \; ; \; \; f = x(1 + r) \]

Value function: \[ V^{CT}(f) = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma(X^f_T-C_T)} \right) \]

Pricing condition: \[ V^0(f) = V^{CT}(f + \nu^f(C_T)) \]

Pricing measure: \[ E_{\mathbb{Q}_f}(F_T) = F_0 \; \text{ and } \; Q^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T) \]

Indifference price: \[ \nu^f(C_T) = E_{\mathbb{Q}_f}(C_T) = E_{\mathbb{Q}_f} \left( \frac{1}{\gamma} \log E_{\mathbb{Q}} \left( e^{\gamma C_T}|F_T \right) \right) \]
Inconsistency across prices expressed in spot and forward units

Pricing measures: \( Q^S = Q^f \)

Spot price: \( \nu^S(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q \left( e^{\gamma \frac{C_T}{1+r}} \big| S_T \right) \right) \)

Forward price: \( \nu^f(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q \left( e^{\gamma C_T} \big| S_T \right) \right) \)

\[ \nu^f(C_T) \neq (1 + r) \nu^s(C_T) \]
What went wrong?

- Risk preferences were not correctly specified!
- Risk aversion is not a constant
- Risk preferences need to be adjusted across units
- Risk preferences cannot be specified in isolation from the market and the investment horizon
Indifference prices in spot and forward units

Spot units

Wealth: \[ X^s_T = x + \alpha \left( \frac{S_T}{1+r} - S_0 \right) \]

Value function: \[ V^{s,C_T}(x) = \sup_{\alpha} \mathbb{E}_\mathbb{P} \left( -e^{-\gamma s \left( X^s_T - \frac{C_T}{1+r} \right)} \right) \]

Pricing condition: \[ V^{s,0}(x) = V^{s,C_T}(x + \nu^s(C_T)) \]

Pricing measure: \[ \mathbb{E}_\mathbb{Q}^s \left( \frac{S_T}{1+r} \right) = S_0 \text{ and } \mathbb{Q}^s(Y_T|S_T) = \mathbb{P}(Y_T|S_T) \]

Indifference price: \[ \nu^s(C_T) = \mathcal{E}_\mathbb{Q}^s \left( \frac{C_T}{1+r} \right) = \mathbb{E}_\mathbb{Q}^s \left( \frac{1}{\gamma s} \log \mathbb{E}_\mathbb{Q}^s \left( e^{\gamma s \frac{C_T}{1+r}} | S_T \right) \right) \]
Forward units

Wealth: \[ X_T^f = X_T^s (1 + r) = f + \alpha (F_T - F_0) ; \quad f = x (1 + r) \]

Value function: \[ V_{f,CT}^f(f) = \sup_{\alpha} E_P \left( -e^{-\gamma f (X_T^f - C_T)} \right) \]

Pricing condition: \[ V_{f,0}^f(f) = V_{f,CT}^f(f + \nu f (C_T)) \]

Pricing measure: \[ E_{Q^f} (F_T) = F_0 \quad \text{and} \quad Q^f (Y_T | F_T) = P (Y_T | F_T) \]

Indifference price: \[ \nu^f (C_T) = E_{Q^f} (C_T) = E_{Q^f} \left( \frac{1}{\gamma} \log E_{Q^f} \left( e^{\gamma f C_T} | F_T \right) \right) \]
Consistency across spot and forward units

\[ \nu^f(C_T) = (1 + r)\nu^s(C_T) \iff \delta^s = \frac{1}{1+r}\delta^f \]

\[ \delta^s = \frac{1}{\gamma^s}, \quad \delta^f = \frac{1}{\gamma^f} : \text{spot and forward risk tolerance} \]

Risk tolerance is not a number. It is expressed in wealth units.
The stock as the numeraire

- Indifference price is a unitless quantity (number of stock shares)

- The “utility argument” $\gamma T \frac{X_T}{S_T}$ needs to be unitless as well

- Static no arbitrage constraint strongly suggests that risk aversion needs to be stochastic
In a bigger picture...

- Semigroup property of prices

\[ \nu_t(C_T; \gamma(\omega)) = \nu_t(\nu_s(C_T; \gamma(\omega)); \gamma(\omega)) \]

- Numeraire invariance

\[ \frac{\nu_t^Y(C_T; \gamma^Y(\omega))}{Y_t} = \nu_t^Z \left( \frac{C_T}{Y_T}; \gamma^Z(\omega) \right) \]
No arbitrage

\[ \uparrow \equiv \downarrow \]

The term structure of risk preferences

\[ \vdash \]

Stochastic utilities
Stochastic risk preferences
Traditional utility and random risk tolerance

- \((\Omega, \mathcal{F}, \mathbb{P}); \mathcal{F}_t^{(S,Y)}\)

  \(S\) : traded security, \(Y\) : non-traded factor

- Wealth process: \(X_s, \quad dX_s = \pi_s \frac{dS_s}{S_s}\)

- Risk preferences normalized at \(T > 0\), expressed in reciprocal to wealth units, via a random variable \(\gamma_T \in \mathcal{F}_T^{(S,Y)}\)
Traditional value function and random risk tolerance

- **Criterion:**
  \[ V(X_t, t) = \sup_{\pi} E_{\mathbb{P}} \left( -e^{-\gamma T X_T} / \mathcal{F}_t \right) \]

- **Time line**

\[
\begin{align*}
V(X_t, t) &\quad V(X_s, s) &\quad -e^{-\gamma T X_T} \\
t &\quad &\quad T
\end{align*}
\]

no information on the "process" \( \gamma_s, t \leq s \leq T \)
Issues

• When do we have (if possible) a wealth affine representation

\[ V(X_s, s) = -e^{-\gamma_s X_s - H_s} \]

\[ \gamma_s \] : risk aversion process

\[ H_s \] : entropy process

• What is the risk aversion process?

• The above needed for efficient indifference pricing
A simple example

One period incomplete binomial model
(MZ 2005)

• Traded and non-traded risks, $S_T$ and $Y_T$

• $\gamma_T = \gamma(S_T)$ $\mathcal{F}_T^S$-measurable random variable
  (in reciprocal to wealth units)

• Risk tolerance (in units of wealth)

  $$\delta_T = \frac{1}{\gamma_T}$$

• Should $\gamma_T$ be allowed to be $\mathcal{F}_T^{(S,Y)}$-measurable?
Random utility and its value function

• Value function

\[ V^0(x; \gamma T) = -\exp \left( -\frac{x}{E_Q(\frac{1}{\gamma T})} - H(Q^* | P) \right) \]

• Value function and utility

\[ V(x, 0; T) = -e^{-\frac{x}{E_Q(\frac{1}{\gamma T})} - H(Q^* | P)} \]

\[ U(X_T; T) = -e^{-\gamma_TX_T} \]
• Two minimal entropy measures

\[
\frac{dQ^*}{dQ} = \frac{\delta_T}{E_Q(\delta_T)}
\]

\[
E_Q(S_T - (1 + r) S_0) = 0
\]

\[
E_{Q^*}(\gamma_T (S_T - (1 + r) S_0)) = 0
\]

• Structural constraints between the market environment and the risk preferences

\[
\gamma_0 = E_Q(\gamma_T / F_0) \quad \leftarrow \ldots \ldots \ldots \ldots \ldots \gamma_T
\]

\[
0 \quad T
\]
How are indifference prices affected by our (non deterministic) views?

- The indifference price of $C_T$ is given by

$$\nu (C_T; \gamma_T; T) = E_Q \left( \frac{1}{\gamma_T} \log E_Q (e^{\gamma_T C_T | S_T}) \right)$$

- Prices depend on the point at which preferences are normalized!

$$U (X_T; T) = - e^{-\gamma_T X_T}$$

- Value function with the claim

$$V^{C_T} (x; \gamma_T) = - \exp \left( - \left( \frac{x - \nu (C_T; \gamma_T)}{E_Q (\delta_T)} \right) - H (Q^* | P) \right)$$
The term structure of risk preferences
**Fundamental questions**

- What is the proper specification of the investors’ risk preferences?
- Are risk preferences static or dynamic?
- Are they affected by the market environment and the trading horizon?
- Are there endogenous structural conditions on risk preferences?
- How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?
Requirements for a well posed indifference pricing system

(work in progress MZ, ZZ)

Risk preferences need to satisfy structural conditions across units, trading horizons and maturities

↓

Self-generating stochastic utilities

Martingality of risk tolerance process is one of the requirements we observed so far
Self-generating stochastic utilities

Utility process:

\[ U_s \quad U_t \quad U_T \quad U_t \in \mathcal{F}_t \]

Value function process:

\[ V_s \quad U_t \quad U_T \quad V_t \in \mathcal{F}_t \]

\[ V_s(X_s; T) = \sup_{A} E_{P} [U_t(X_t; T)/\mathcal{F}_s] \]

Requirement for correct pricing

\[ V_s = U_s \quad V_t = U_t \quad V_t = U_T \]

Alignment of current utility with its associated value function
Self-generation of dynamic utility

- No static arbitrage
- Consistency of prices across units
- Semigroup property of prices
- Invariance, under optimal investment behavior, with respect to trading horizons and maturities
Models of the term structure of risk preferences

- Normalization point
- Evolution equation

Questions

- Are dynamic risk preferences and market behavior interlinked?
- How is the utility evolving w.r.t. time?

Answers

- Preferences and market cannot be defined in isolation
- Utilities are defined backward and forward in time
Backward stochastic utilities

- Traded asset: \( S_t, \ t \in [0, T] \), Non-traded asset: \( Y_t, \ t \in [0, T] \)

Filtrations: \( \mathcal{F}^S_t, \ \mathcal{F}^Y_t, \ \mathcal{F}^{(S,Y)}_t \)

- Normalization point: \( T > 0 \)

- The backward, normalized at \( T > 0 \), dynamic utility \( U^B_t(x; T) \) is defined as an \( \mathcal{F}^{(S,Y)}_t \)-measurable process solving

\[
U^B_S(X_s; T) = \sup_{\alpha} E_{\mathbb{P}} \left( U^B_t(X_t; T) | \mathcal{F}^{(S,Y)}_s \right)
\]

\[
X_t = X_s + \int_s^t \alpha_u \, dS_u
\]

\[
U^B(x, T; T) = U(x; \gamma_T) ; \ U \text{ given}
\]
Backward stochastic utilities

\[ U_s^B(X_s; T) = \sup_{\mathcal{A}} E[U_t^B(X_t; T)/\mathcal{F}_s] = \sup_{\mathcal{A}} E[U_T^B(X_T; T)/\mathcal{F}_s] = V_s(X_s; T) \]

- Self-generation and invariance w.r.t. trading horizons
- Backward utility is, essentially, the traditional value function
- Backward utility and market behavior interlinked
Complete market and stochastic risk tolerance

Example

\[ U_T^B = -e^{-\gamma_T X_T} \]

\[ U_s^B(X_s; T) = \sup_{\mathcal{A}} E_P \left[ U_t^B(X_t; T)/\mathcal{F}_s \right] = -e^{-\frac{X_s}{\delta_s} - \tilde{H}_s} \]

\[ \delta_s = E_Q \left( \frac{1}{\gamma_T}/\mathcal{F}_s \right) \quad \tilde{H}_s = E_{\tilde{Q}} \left[ \int_s^T \frac{1}{2} \lambda_u^2 du / \mathcal{F}_s \right] \]

\[ S_s = E_Q(S_T/\mathcal{F}_s) \quad \frac{1}{\delta_s} S_s = E_{\tilde{Q}} \left( \frac{1}{\delta_T} S_t/\mathcal{F}_s \right) \]
Forward stochastic utility (MZ 2005)

• How do we decide today about our future risk attitude?

• How do we valuate, via utility, claims of arbitrary maturities?

\[
\text{fixed horizon } T \\
\begin{array}{c}
t \\
C_s, \ldots, C_T \\
\end{array}
\]

\[
\text{valuation already done w.r.t. normalization point } T
\]

\[
\text{new claim arrives}
\]

\[
T
\]

\[
T'
\]

\[
\text{longer maturity}
\]
Forward stochastic utility (con’)

• Normalization point $s > 0$

• Dynamic utility is specified going \textit{forward in time}, $t \geq s$

• \textbf{No need} to have a \textbf{fixed} trading horizon
Forward stochastic utility

- **Normalization point:** $s > 0$

- $U^F_t(x; s), s > 0, t \geq s$ is defined as an $\mathcal{F}_t^{(S,Y)}$-measurable process solving

\[
U^F_t(X_t; s) = \sup_\alpha E_\mathbb{P}(U^F_{t'}(X_t; s)|\mathcal{F}_t^{(S,Y)})
\]

\[
X_{t'} = X_t + \int_t^{t'} \alpha_u \, dS_u
\]

\[
U^F_s(x) = U(x; \omega) ; \quad U \text{ given}
\]
Forward stochastic utilities

\[ U^F_s(x; s) \]

\[ 0 \quad s \quad t \quad t' \quad +\infty \]

utility generation forward in time

\[ U^F_t(X_t; s) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left[ U^F_{t'}(X_{t'}; s)/\mathcal{F}_s \right] \]

\[ = V^F_t(X_t; s) \]

Self-generation and invariance w.r.t. trading horizon
Complete market case and stochastic Sharpe ratio

Example

\[ U_s^F(x; s) = -e^{-\gamma x + \int_s^t \frac{1}{2} \lambda_u^2 du} \]

Self-generation

\[ -e^{-\gamma x + \int_s^t \frac{1}{2} \lambda_u^2 du} = \sup_{\mathcal{A}} E_{\mathbb{P}} \left[ -e^{-\gamma X_{t'} + \int_t^{t'} \frac{1}{2} \lambda_u^2 du} / \mathcal{F}_t \right] \]
Forward versus backward utilities

- **Backward** stochastic utilities aggregate market information while **forward** stochastic utilities use information revealed *dynamically* by the market.

- **Forward** indifference prices do not depend on the preference normalization point.

- Forward indifference prices are represented in a *more intuitive* manner.
Open questions

• When do backward stochastic utilities exist? Are they unique? Are they robust w.r.t. terminal risk preference specification? (relatively easy)

• When do forward stochastic utilities exist? Are they unique? Are they robust w.r.t. initial risk preference specification? (very hard)

• What is the term structure of the risk tolerance process in backward and forward setting?

• How is indifference valuation built in terms of backward and forward preferences?

• Are backward and forward indifference prices equal?

• How do they depend on the risk preference normalization point?
Results to date

Backward and forward stochastic utilities and their associated prices and hedges

- Binomial model
- Diffusion model
- Stochastic risk tolerance

Exponential preferences

Power preferences (partial results)
Utility-based backward and forward pricing systems for stochastic volatility models
The diffusion case

• Market dynamics

\[ dS_s = \mu(Y_s, s)S_s \, ds + \sigma(Y_s, s)S_s \, dW^1_s \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]

\[ \rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

• Minimal relative entropy measure:

\[
\frac{dQ}{dP} = \exp \left( - \int_0^T \lambda_s \, dW^1_s - \int_0^T \lambda^1_s \, dW^1_s, \perp - \frac{1}{2} \int_0^T (\lambda^2_s + (\lambda^1_s)^2) \, ds \right) \\
\lambda^\perp_s = \lambda^\perp_s(Y_s, s); \quad \lambda^\perp(y, t) \sim \text{gradient to the sln of a quasilinear pde} \\
\text{(Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)}
\]
Backward stochastic utilities

For $T > 0$ the process $\{U^B_t(x; T) : 0 \leq t \leq T\}$ defined, for $x \in \mathbb{R}$, by

$$U^B_t(x; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x - H_T(Q(\cdot|\mathcal{F}_t)|P(\cdot|\mathcal{F}_t))} & \text{if } 0 \leq t < T 
\end{cases}$$

with

$$H_T(Q(\cdot|\mathcal{F}_t)|P(\cdot|\mathcal{F}_t)) = E_Q \left( \int^T_t \frac{1}{2} (\lambda_s^2 + (\lambda^\perp_s)^2) \, ds \mid \mathcal{F}_t \right)$$

is the, normalized at time $T$, backward stochastic exponential utility.
Forward stochastic utilities \hfill (MZ 2005)

For $T \geq s \geq 0$ and $x \in \mathbb{R}$, the process $U_t^F(x; s) e^{\mathcal{F}_t^{(S,Y)}}$ defined by

$$U_t^F(x; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x} + h_t & \text{if } t \geq s
\end{cases}$$

with

$$h_t = \int_s^t \frac{1}{2} \lambda^2(Y_u, u) \, du$$

is the forward, normalized at time $s$, stochastic exponential utility.
Backward and forward indifference prices

• Backward price

\[ h_t^B (C_{\hat{T}}; T) = \mathcal{E}_{Q_{mm}}^{(t, \hat{T})} (C_{\hat{T}}; T) \]

\( \hat{T} \leq T, \ Q_{mm} : \) minimal martingale measure

• Forward exponential utility

\[ h_t^F (C_{\hat{T}}; s) = \mathcal{E}_{Q_{me}}^{(t, \hat{T})} (C_{\hat{T}}) \]

No dependence on \( s, \ Q_{me} : \) minimal entropy measure
Before we “become indifferent” and quote prices, we need to understand

- how we formulate our risk appetite
- how our risk tolerance and the market environment affect each other
- how we built our dynamic utility structure

**Technical challenges**

- Stochastic pdes
- Forward utility might give rise to “not well-posed” problems